

# Local Langlands correspondence and ramification for Carayol representations

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**ABSTRACT.** Let  $F$  be a non-Archimedean locally compact field of residual characteristic  $p$  with Weil group  $\mathcal{W}_F$ . Let  $\sigma$  be an irreducible smooth complex representation of  $\mathcal{W}_F$ , realized as the Langlands parameter of an irreducible cuspidal representation  $\pi$  of a general linear group over  $F$ . In an earlier paper, we showed that the ramification structure of  $\sigma$  is determined by the fine structure of the endo-class  $\Theta$  of the simple character contained in  $\pi$ , in the sense of Bushnell-Kutzko. The connection is made via the *Herbrand function*  $\Psi_\Theta$  of  $\Theta$ . In this paper, we concentrate on the fundamental Carayol case in which  $\sigma$  is totally wildly ramified with Swan exponent not divisible by  $p$ . We show that, for such  $\sigma$ , the associated Herbrand function satisfies a certain symmetry condition or functional equation, a property that essentially characterizes this class of representations. We calculate  $\Psi_\Theta$  explicitly, in terms of a classical Herbrand function coming from the classification of simple characters. We describe exactly the class of functions arising as Herbrand functions  $\Psi_\Xi$ , as  $\Xi$  varies over the set of totally wild endo-classes of Carayol type. In a separate argument, we get a complete description of  $\sigma$  restricted to any ramification subgroup. This provides a different, more Galois-centred, view on the Herbrand function.

**1.** Let  $F$  be a non-Archimedean, locally compact field with residual characteristic  $p$ . Let  $\mathcal{W}_F$  be the Weil group of a separable closure  $\bar{F}/F$ . For a real variable  $x \geq 0$ , let  $\mathcal{R}_F(x) = \mathcal{W}_F^x$  be the corresponding ramification subgroup of  $\mathcal{W}_F$  and  $\mathcal{R}_F^+(x)$  the closure of  $\bigcup_{y>x} \mathcal{R}_F(y)$ . We use the conventions of [37] here, so that  $\mathcal{R}_F(0)$  is the inertia group  $\mathcal{I}_F$  and  $\mathcal{R}_F^+(0)$  is the wild inertia group  $\mathcal{P}_F$  in  $\mathcal{W}_F$ . If  $\mathcal{G}$  is any of this list of locally profinite groups,  $\hat{\mathcal{G}}$  will denote the set of equivalence classes of irreducible, smooth, complex representations of  $\mathcal{G}$ . We

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shall be concerned with the *ramification structure* of certain  $\sigma \in \widehat{\mathcal{W}}_F$ , that is, the structure of the restrictions  $\sigma|_{\mathcal{R}_F(x)}$ ,  $\sigma|_{\mathcal{R}_F^+(x)}$ ,  $x > 0$ .

On the other side, let  $\mathcal{A}_n^0(F)$  denote the set of equivalence classes of irreducible, cuspidal, complex representations of the general linear group  $\mathrm{GL}_n(F)$ ,  $n \geq 1$ , and set  $\widehat{\mathrm{GL}}_F = \bigcup_{n \geq 1} \mathcal{A}_n^0(F)$ . For  $\pi \in \widehat{\mathrm{GL}}_F$ , we write  $\mathrm{gr}(\pi) = n$  to indicate  $\pi \in \mathcal{A}_n^0(F)$ . Such a representation  $\pi$  contains a *simple character*  $\theta_\pi$  in  $\mathrm{GL}_n(F)$  [15] and, up to conjugation, only one [10]. The *endo-class*  $\Theta_\pi$  of  $\theta_\pi$  is therefore uniquely determined by  $\pi$ . (For the notion of endo-class, see [3] or the summary in any of [2, 5, 10].) Let  $\mathcal{E}(F)$  denote the set of endo-classes of simple characters over  $F$ .

Denote by  $\pi \mapsto {}^L\pi$  the Langlands correspondence  $\widehat{\mathrm{GL}}_F \rightarrow \widehat{\mathcal{W}}_F$  [19, 24, 34, 36]. Writing  $\sigma = {}^L\pi$ , we showed in [13] 6.5 Corollary that the fine structure of the endo-class  $\Theta_\pi$  and the ramification structure of  $\sigma$  determine each other. In this paper, we investigate the relation for a class of representations of fundamental importance, the so-called Carayol representations. For this class, we show how to compute the Herbrand function of an endo-class, we list the functions which arise as Herbrand functions and we interpret the function in terms of the ramification structure of the associated Galois representation.

**2.** We review the background from [13] with as little formality as possible. If  $\pi \in \widehat{\mathrm{GL}}_F$  and  $\sigma = {}^L\pi \in \widehat{\mathcal{W}}_F$ , the endo-class  $\Theta_\pi$  determines the restriction  $\sigma|_{\mathcal{P}_F}$ . More precisely,  $\sigma$  defines an element  $[\sigma]_0^+$  of the orbit space  $\mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$ , namely the orbit of irreducible components of  $\sigma|_{\mathcal{P}_F}$ . The Langlands correspondence then induces a canonical bijection

$$(A) \quad \begin{aligned} \mathcal{E}(F) &\longrightarrow \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F, \\ \Theta &\longmapsto {}^L\Theta, \end{aligned}$$

by

$$[{}^L\pi]_0^+ = {}^L\Theta_\pi, \quad \pi \in \widehat{\mathrm{GL}}_F.$$

(See [5] 8.2 Theorem, [12] 6.1.)

The starting point of [13] is that each of the sets  $\mathcal{E}(F)$ ,  $\mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$  carries a canonical *ultrametric*. That on  $\mathcal{E}(F)$ , denoted by  $\mathbb{A}$ , is built on the fact that simple characters are characters of compact groups carrying canonical filtrations, and those filtrations provide a medium via which the characters may be compared. The ultrametric  $\mathbb{A}$  relates to Swan exponents of pairs of representations, as defined from the local constants of [28, 38]. Let  $\Theta \in \mathcal{E}(F)$  and choose

$\pi \in \widehat{\mathrm{GL}}_F$  such that  $\Theta_\pi = \Theta$ . There is a unique continuous function  $\Phi_\Theta(x)$ ,  $x \geq 0$ , such that

$$\Phi_\Theta(\mathbb{A}(\Theta, \Theta_\rho)) = \frac{\mathrm{sw}(\check{\pi} \times \rho)}{\mathrm{gr}(\pi) \mathrm{gr}(\rho)},$$

for any  $\rho \in \widehat{\mathrm{GL}}_F$ . The function  $\Phi_\Theta$  is piecewise linear, strictly increasing and convex. It is given by an explicit formula [13] (4.4.1) derived from the conductor formula of [14] 6.5 Theorem. We call  $\Phi_\Theta$  the *structure function* of  $\Theta$ .

The ultrametric on  $\mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$ , denoted by  $\Delta$ , is defined by comparing representations via the canonical filtration of  $\mathcal{P}_F$  by ramification groups: for  $\sigma, \tau \in \widehat{\mathcal{W}}_F$ ,

$$\Delta([\sigma]_0^+, [\tau]_0^+) = \inf\{x > 0 : \mathrm{Hom}_{\mathcal{R}_F(x)}(\sigma, \tau) \neq 0\}.$$

The ultrametric  $\Delta$  likewise relates to Swan exponents of tensor products of pairs of representations of  $\mathcal{W}_F$  [20]. For  $\sigma \in \widehat{\mathcal{W}}_F$ , there is a unique continuous function  $\Sigma_\sigma(x)$ ,  $x \geq 0$ , such that

$$\Sigma_\sigma(\Delta([\sigma]_0^+, [\tau]_0^+)) = \frac{\mathrm{sw}(\check{\sigma} \otimes \tau)}{\dim \sigma \cdot \dim \tau},$$

for all  $\tau \in \widehat{\mathcal{W}}_F$ . The function  $\Sigma_\sigma$  is piecewise linear, strictly increasing and convex. It is given by a formula derived from the ramification structure of  $\sigma$ , reproduced in (2.2.2) below. If  $\Sigma_\sigma$  is smooth at  $x$ , its derivative satisfies

$$\Sigma'_\sigma(x) = \dim \mathrm{End}_{\mathcal{R}_F(x)}(\sigma) / (\dim \sigma)^2.$$

We call  $\Sigma_\sigma$  the *decomposition function* of  $\sigma$ : it depends only on the orbit  $[\sigma]_0^+$ .

If  $\Theta \in \mathcal{E}(F)$ , set  $\Psi_\Theta = \Phi_\Theta^{-1} \circ \Sigma_\sigma$ , for any  $\sigma \in \widehat{\mathcal{W}}_F$  such that  $[\sigma]_0^+ = {}^L\Theta$ . The Langlands correspondence respects Swan exponents of pairs and  $\dim {}^L\pi = \mathrm{gr}(\pi)$ ,  $\pi \in \widehat{\mathrm{GL}}_F$ , so

$$\Psi_\Theta(\Delta({}^L\Theta, {}^L\Xi)) = \mathbb{A}(\Theta, \Xi), \quad \Xi \in \mathcal{E}(F).$$

The function  $\Psi_\Theta$  is called the *Herbrand function* of  $\Theta$ . It is continuous, strictly increasing and piecewise linear.

If we take the view that  $\Theta \in \mathcal{E}(F)$  has been given, in terms of the standard classification from [15], it is a simple matter to write down the function  $\Phi_\Theta$ . The Interpolation Theorem [13] 7.5 shows, in principle, how to compute  $\Psi_\Theta$  *directly* from  $\Theta$ , without reference to  ${}^L\Theta$ . This yields the decomposition function  $\Sigma_\sigma$  and therefore a numerical account of the ramification structure of  $\sigma$ , just in terms of  $\Theta$ .

**3.** Let  $\Theta \in \mathcal{E}(F)$ . Assuming, as we invariably do, that  $\Theta$  is non-trivial, it is the endo-class of a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  attached to a simple stratum  $[\mathfrak{a}, m, 0, \beta]$  in some matrix algebra  $M_n(F)$  (following the conventions of [15]). In particular,  $\beta \in \mathrm{GL}_n(F)$  and the algebra  $F[\beta]$  is a field. The positive integers  $\deg \Theta = [F[\beta]:F]$  and  $e_\Theta = e(F[\beta]|F)$  are invariants of  $\Theta$ . The *slope*  $\varsigma_\Theta$  of  $\Theta$ , defined by  $\varsigma_\Theta = m/e_\mathfrak{a}$ , where  $e_\mathfrak{a}$  is the period of the hereditary  $\mathfrak{o}_F$ -order  $\mathfrak{a}$ , is likewise an invariant of  $\Theta$ . If  $\pi \in \widehat{\mathrm{GL}}_F$  satisfies  $\Theta_\pi = \Theta$ , then  $\varsigma_\Theta = \mathrm{sw}(\pi)/\mathrm{gr}(\pi)$ . Be aware, however, that neither  $\theta$  nor  $\Theta$  determines the field  $F[\beta]$ , even up to isomorphism.

Say that  $\Theta \in \mathcal{E}(F)$  is *totally wild* if  $\deg \Theta = e_\Theta = p^r$ , for an integer  $r \geq 0$ . If  $\Theta$  is totally wild, say it is *of Carayol type* if  $\deg \Theta > 1$  and the integer  $e_\Theta \varsigma_\Theta$  is not divisible by  $p$ . Let  $\mathcal{E}^C(F)$  denote the set of endo-classes  $\Theta \in \mathcal{E}(F)$  that are totally wild of Carayol type. We aim to calculate  $\Psi_\Theta$  for all  $\Theta \in \mathcal{E}^C(F)$ .

We concentrate on this case for two reasons. First, 7.1 Proposition of [13] reduces the problem of calculating Herbrand functions to the totally wild case (see (2.2.3) below). Second, we have to work with simple characters. The definition of simple character in [15] is rigidly hierarchical in nature and proofs are almost always inductive along this hierarchy. The first inductive step concerns the case where the element  $\beta$  (as above) is *minimal over  $F$*  [15] (1.4.14). For totally wild endo-classes, this is the Carayol case. (The name is given in recognition of Carayol's seminal paper [17], where the case was first systematically investigated and given due prominence.)

Further motivation comes from the Galois side. If  $\sigma \in \widehat{\mathcal{W}}_F$ , say that  $\sigma$  is *totally wild* if the restriction  $\sigma|_{\mathcal{P}_F}$  is irreducible. In particular,  $\dim \sigma = p^r$ , for some  $r \geq 0$ . Denote by  $\widehat{\mathcal{W}}_F^{\mathrm{wr}}$  the set of totally wild elements of  $\widehat{\mathcal{W}}_F$ . An endo-class  $\Theta \in \mathcal{E}(F)$  is then totally wild if and only if there exists  $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$  such that  $[\sigma]_0^+ = {}^L\Theta$  (cf. section 6 of [12]). In a series of papers [3, 4, 5, 6, 7, 9] and particularly [12], we have reduced the problem of giving an explicit description of the Langlands correspondence  $\widehat{\mathrm{GL}}_F \rightarrow \widehat{\mathcal{W}}_F$  to the case of totally wild representations. Indeed, the main results of [12] show how to reconstruct the correspondence from the bijection (A), *restricted to totally wild endo-classes*. A better understanding of that map thereby becomes a priority.

Say that  $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$  is *of Carayol type* if  $\dim \sigma \neq 1$  and  $p$  does not divide  $\mathrm{sw}(\sigma)$ . Thus  $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$  is of Carayol type if and only if  $[\sigma]_0^+ = {}^L\Theta$ , for some  $\Theta \in \mathcal{E}^C(F)$ . In this context, the implications for  $\sigma$  of the hypothesis  $p \nmid \mathrm{sw}(\sigma)$  are not immediately apparent. As a consequence of this investigation, we shall see that they are profound.

4. We review our principal results. These are organized into three main theorems, complementing and supporting each other.

For any  $\Theta \in \mathcal{E}(F)$ , the Herbrand function  $\Psi_\Theta(x)$  satisfies  $\Psi_\Theta(0) = 0$  and  $\Psi_\Theta(x) = x$  for  $x \geq \varsigma_\Theta$  [13] 6.2 Proposition. The derivative  $\Psi'_\Theta(x)$  has only finitely many discontinuities in the interesting region  $0 < x < \varsigma_\Theta$ : we call these the *jumps* of  $\Psi_\Theta$ . When  $\Theta \in \mathcal{E}^C(F)$ , the function  $\Psi_\Theta(x)$  is *convex* in the region  $0 \leq x \leq \varsigma_\Theta$ . The reasons are simple (see 2.4), but the property is very useful.

**Theorem 1.** *Let  $\Theta \in \mathcal{E}^C(F)$ . The graph  $y = \Psi_\Theta(x)$ ,  $0 \leq x \leq \varsigma_\Theta$ , is symmetric with respect to the line  $x+y = \varsigma_\Theta$ . That is,*

$$(B) \quad \varsigma_\Theta - x = \Psi_\Theta(\varsigma_\Theta - \Psi_\Theta(x)), \quad 0 \leq x \leq \varsigma_\Theta.$$

Theorem 1 has a satisfying converse. The group of characters of  $U_F^1$  acts on the set  $\mathcal{E}(F)$  following the natural twisting action of characters of  $F^\times$  or  $\mathcal{W}_F$  on  $\widehat{\text{GL}}_F$  or  $\widehat{\mathcal{W}}_F$ . We denote this action by  $(\chi, \Theta) \mapsto \chi\Theta$ . By 7.4 Proposition of [13], it has the property  $\Psi_{\chi\Theta} = \Psi_\Theta$ . We get:

**Corollary.** *Let  $\Theta \in \mathcal{E}(F)$  be totally wild of degree  $p^r$ , for some  $r \geq 1$ , and suppose that  $\varsigma_\Theta \leq \varsigma_{\chi\Theta}$  for all characters  $\chi$  of  $U_F^1$ . The function  $\Psi_\Theta$  then has the symmetry property (B) if and only if  $\Theta \in \mathcal{E}^C(F)$ .*

Theorem 1, together with some preliminary calculations, suggests the definition of a family of elementary functions. Let  $r \geq 1$  and let  $E/F$  be a totally ramified field extension of degree  $p^r$ . Let  $m$  be a positive integer not divisible by  $p$  and set  $\varsigma = m/p^r$ . Let  $\psi_{E/F}$  be the classical Herbrand function of  $E/F$  [18, 37]. Define  $c$  by the equation  $c + p^{-r}\psi_{E/F}(c) = \varsigma$ . There is then a unique function  ${}^2\Psi_{(E/F, \varsigma)}(x)$ ,  $0 \leq x \leq \varsigma$ , such that the graph  $y = {}^2\Psi_{(E/F, \varsigma)}(x)$  is symmetric with respect to the line  $x+y = \varsigma$  and  ${}^2\Psi_{(E/F, \varsigma)}(x) = p^{-r}\psi_{E/F}(x)$ , for  $0 \leq x \leq c$ . Functions of this form will be called *bi-Herbrand functions*.

Our strategy is to identify  $\Psi_\Theta$ ,  $\Theta \in \mathcal{E}^C(F)$ , as a specific bi-Herbrand function. Let  $\deg \Theta = p^r$ . There is a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M_{p^r}(F)$  such that  $\Theta$  is the endo-class of some  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ . This condition ensures that  $F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $p$  does not divide  $m = -v_{F[\alpha]}(\alpha)$ . Here,  $\varsigma_\Theta = m/p^r$ . Let  $\|\mathcal{C}(\mathfrak{a}, \alpha)\|$  be the set of endo-classes of elements of  $\mathcal{C}(\mathfrak{a}, \alpha)$ . We then have  $\|\mathcal{C}(\mathfrak{a}, \alpha)\| \subset \mathcal{E}^C(F)$ .

The set  $\|\mathcal{C}(\mathfrak{a}, \alpha)\|$  is not well-adapted to our purposes, because the function  $\Theta \mapsto \Psi_\Theta$  is not constant there. Indeed, it may vary quite widely (see 7.2

Theorem 1). To overcome this problem, we specify a non-empty subset  $\mathcal{C}^*(\mathfrak{a}, \alpha)$  of  $\mathcal{C}(\mathfrak{a}, \alpha)$ , using an explicit formula given in 7.1 below. Let  $\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$  denote the set of endo-classes of characters  $\theta \in \mathcal{C}^*(\mathfrak{a}, \alpha)$ .

**Theorem 2.** *Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$  and  $\varsigma_\Theta = m/p^r$ . There is a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M_{p^r}(F)$  such that  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ . For any such stratum,*

$$(C) \quad \Psi_\Theta(x) = {}^2\Psi_{(F[\alpha]/F, \varsigma_\Theta)}(x), \quad 0 \leq x \leq \varsigma_\Theta.$$

Theorem 2 has the following consequence.

**Corollary.** *Let  $E/F$  be a totally ramified field extension of degree  $p^r$ ,  $r \geq 1$ , and let  $m$  be a positive integer not divisible by  $p$ . There exists  $\Theta \in \mathcal{E}^C(F)$ , of degree  $p^r$ , such that*

$$\Psi_\Theta(x) = {}^2\Psi_{(E/F, m/p^r)}(x), \quad 0 \leq x \leq m/p^r = \varsigma_\Theta.$$

The corollary is an effective tool for constructing representations of  $\mathcal{W}_F$  with specified ramification properties. An example of the technique is given in 9.7.

**5.** In our third result, we look at the problem from the Galois side. Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type and dimension  $p^r$ . Define  $\Theta \in \mathcal{E}^C(F)$  by  $[\sigma]_0^+ = {}^L\Theta$ . As  $r \geq 1$ , the function  $\Psi_\Theta$  has at least one jump [13] 7.7. If  $\Psi_\Theta$  has *exactly* one jump, we say that  $\sigma$  is *H-singular*. We will analyze these representations carefully: they belong to a rather special class of “Heisenberg representations” (as one says).

Without restriction on the number of jumps, define a number  $c_\Theta$  by the equation

$$c_\Theta + \Psi_\Theta(c_\Theta) = \varsigma_\Theta, \quad \Theta \in \mathcal{E}^C(F).$$

By symmetry,  $c_\Theta$  is a jump of  $\Psi_\Theta$  if and only if  $\Psi_\Theta$  has an odd number of jumps and, in that case,  $c_\Theta$  is the middle one.

**Theorem 3.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type and dimension  $p^r$ . Let  $\Theta \in \mathcal{E}^C(F)$  satisfy  ${}^L\Theta = [\sigma]_0^+$ .*

- (1) *The restriction  $\sigma|_{\mathcal{R}_F^+(c_\Theta)}$  is a direct sum of characters.*
- (2) *Let  $\xi$  be a character of  $\mathcal{R}_F^+(c_\Theta)$  occurring in  $\sigma$ , let  $\mathcal{W}_{L_\xi}$  be the  $\mathcal{W}_F$ -stabilizer of  $\xi$ , and let  $\sigma_\xi$  be the natural representation of  $\mathcal{W}_{L_\xi}$  on the*

$\xi$ -isotypic subspace of  $\sigma|_{\mathcal{R}_F^+(c_\Theta)}$ . The field extension  $L_\xi/F$  is totally ramified of degree dividing  $p^r$  and  $\sigma = \text{Ind}_{L_\xi/F} \sigma_\xi$ . Moreover,

- (D)  $\Psi_\Theta(x) = p^{-r} \psi_{L_\xi/F}(x), \quad 0 \leq x \leq c_\Theta.$
- (3) If  $\Psi_\Theta$  has an odd number of jumps, then  $\sigma_\xi$  is irreducible, totally wild,  $H$ -singular, of Carayol type and dimension  $p^r/[L_\xi:F] \neq 1$ .
- (4) If  $\Psi_\Theta$  has an even number of jumps, then  $\sigma_\xi$  is a character and  $[L_\xi:F] = p^r$ .

By symmetry, the relation (D) determines  $\Psi_\Theta$  completely. Any two choices of the character  $\xi$  are  $\mathcal{W}_F$ -conjugate, so the same applies to the field  $L_\xi$ . The field extension  $L_\xi/F$  is not usually Galois but, after a suitable tamely ramified base field extension, it has a canonical presentation as a tower of elementary abelian extensions faithfully reflecting the ramification structure of  $\sigma$ .

The canonical presentation of  $\sigma$  as an induced representation,

$$\sigma = \text{Ind}_{L_\xi/F} \sigma_\xi = \text{Ind}_{\mathcal{W}_{L_\xi}}^{\mathcal{W}_F} \sigma_\xi,$$

is derived from arithmetic considerations. It can claim to be more natural than anything provided by a purely group-theoretic approach.

The restrictions  $\sigma|_{\mathcal{R}_F(x)}$ ,  $\sigma|_{\mathcal{R}_F^+(x)}$  follow a clear pattern, underlying the symmetry property of Theorem 1. The details are in 9.2 Complement 1 and 9.5 Complement. To give the flavour, suppose there are at least two jumps. Let  $j$  be the least and  $\bar{j}$  the greatest. The restriction  $\sigma|_{\mathcal{R}_F(j)}$  is irreducible, while  $\sigma|_{\mathcal{R}_F^+(\bar{j})}$  is a multiple of a character. The restriction  $\sigma|_{\mathcal{R}_F^+(j)}$  is a multiplicity-free direct sum of irreducible representations while  $\sigma|_{\mathcal{R}_F(\bar{j})}$  is a direct sum of characters, its isotypic components being the restrictions of the irreducible components of  $\sigma|_{\mathcal{R}_F^+(j)}$ . The pattern repeats for the second and penultimate jump, and so on.

**6.** We now have two expressions, (C) and (D), for the Herbrand function  $\Psi_\Theta$  of  $\Theta \in \mathfrak{E}^C(F)$ . Together they show how to read the *algebraic* structure of the decompositions  $\sigma|_{\mathcal{R}_F(x)}$ ,  $x > 0$ , directly from the presentation  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ . Our final tranche of results considers more arithmetic aspects.

In the same context, the number  $c_\Theta$  (as in part 5) is constant as  $\Theta$  ranges over  $\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ . We therefore denote it  $c_\alpha$ . Let  $j_\infty = j_\infty(F[\alpha]|F)$  be the largest jump of the classical Herbrand function  $\psi_{F[\alpha]/F}$ . The definition of  ${}^2\Psi_{(F[\alpha]/F, \varsigma_\Theta)}$  and Theorem 2 show that  $\Psi_\Theta$  has an even number of jumps if and only if  $j_\infty < c_\alpha$ .

Let  $\mathcal{G}^*(\alpha)$  be the set of  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  such that  $[\sigma]_0^+ \in {}^L\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ .

**Theorem 4A.** *If  $\sigma_1, \sigma_2 \in \mathcal{G}^*(\alpha)$ , the representations  $\sigma_1|_{\mathcal{R}_F^+(c_\alpha)}$ ,  $\sigma_2|_{\mathcal{R}_F^+(c_\alpha)}$  are equivalent. In particular, any character  $\xi$  of  $\mathcal{R}_F^+(c_\alpha)$  occurring in  $\sigma_1|_{\mathcal{R}_F^+(c_\alpha)}$  also occurs in  $\sigma_2|_{\mathcal{R}_F^+(c_\alpha)}$ .*

In other words, all representations  $\sigma \in \mathcal{G}^*(\alpha)$  give rise to the same conjugacy class of field extensions  $L_\xi/F$ .

Consider the simpler case in which  $j_\infty(F[\alpha]|F) < c_\alpha$ . The function  $\Psi_\Theta$  then has an even number of jumps. Choose a character  $\xi$  of  $\mathcal{R}_F^+(c_\alpha)$  occurring in some, hence any,  $\sigma \in \mathcal{G}^*(\alpha)$  and abbreviate  $L = L_\xi$ . We then get a bijection  $\chi \mapsto \text{Ind}_{L/F} \chi$  between the set of characters  $\chi$  of  $\mathcal{W}_L$ , agreeing with  $\xi$  on the subgroup  $\mathcal{R}_F^+(c_\alpha)$  of  $\mathcal{W}_L$ , and the set  $\mathcal{G}^*(\alpha)$ . In this case,  $\psi_{F[\alpha]/F} = \psi_{L/F}$ .

In the more complicated case  $j_\infty \geq c_\alpha$ , there is a second field extension to be taken into account. If  $\tau \in \widehat{\mathcal{W}}_F$  has dimension  $n$ , let  $\bar{\tau} : \mathcal{W}_F \rightarrow \text{PGL}_n(\mathbb{C})$  be the associated projective representation. The kernel of  $\bar{\tau}$  is of the form  $\mathcal{W}_E$ , where  $E/F$  is finite and Galois. One calls  $E/F$  the *centric field* of  $\tau$ . Returning to the main topic, let  $\tilde{L}_{\sigma,\xi}/L_\xi$  be the centric field of the H-singular representation  $\sigma_\xi \in \widehat{\mathcal{W}}_{L_\xi}^{\text{wr}}$ . The extension  $\tilde{L}_{\sigma,\xi}/L_\xi$  is non-trivial and Galois. Indeed, the  $p$ -part of the ramification index  $e(\tilde{L}_{\sigma,\xi}|L_\xi)$  is  $(\dim \sigma_\xi)^2$ .

Let  $w_\alpha = w_{F[\alpha]/F}$  be the wild exponent (1.6.1) of the field extension  $F[\alpha]/F$ .

**Theorem 4B.** *Let  $\sigma_1, \sigma_2 \in \mathcal{G}^*(\alpha)$  and suppose  $j_\infty \geq c_\alpha$ .*

- (1) *If  $j_\infty > c_\alpha$ , the representations  $\sigma_1, \sigma_2$  are then  $\mathcal{R}_F(c_\alpha)$ -equivalent and  $\tilde{L}_{\sigma_1,\xi} = \tilde{L}_{\sigma_2,\xi}$ .*
- (2) *Suppose  $j_\infty = c_\alpha$ . If  $T_{\sigma_i,\xi}/L_\xi$  is the maximal tame sub-extension of  $\tilde{L}_{\sigma_i,\xi}/L_\xi$ , then  $T_{\sigma_1,\xi} = T_{\sigma_2,\xi}$ . The fields  $\tilde{L}_{\sigma_1,\xi}, \tilde{L}_{\sigma_2,\xi}$  are equal except possibly in the case where*
  - (a) *the integer  $l_\alpha = m - w_\alpha$  is positive and even, and*
  - (b)  *$\mathbb{A}(\Theta_1, \Theta_2) = l_\alpha/2p^r$ .*

(We remark that, for any  $\Theta_1, \Theta_2 \in \|\mathcal{C}^*(\mathbf{a}, \alpha)\|$ , the number  $p^r \mathbb{A}(\Theta_1, \Theta_2)$  is an integer, at most  $l_\alpha/2$ .) In the exceptional case of part (2), we estimate the number of conjugacy classes of fields  $\tilde{L}_{\sigma,\xi}$  as  $\sigma$  ranges over  $\mathcal{G}^*(\alpha)$ . There is an absolute bound of  $(\dim \sigma_\xi)^2$ , achieved when  $T_{\sigma,\xi} = L_\xi$ .

We say nothing of the relation between the fields  $F[\alpha]$  and  $L_\xi, \tilde{L}_{\sigma,\xi}$ . Even in dimension  $p$ , this is a challenging problem. In the present more general context, it would constitute a major project. We confine ourselves to a brief literature survey at the end of section 10.



7. We give an overview of our methods and the layout of the paper.

Section 1 is a free-standing account of the classical Herbrand functions  $\psi_{E/F}$ ,  $\varphi_{E/F}$  of a finite field extension  $E/F$ . For Galois extensions  $E/F$ , most of what we need can be readily deduced from the standard account in [37]. We develop the same level of detail for non-Galois extensions, starting from Deligne's notes [18]. It is convenient to include inseparable extensions here, the effort being negligible. This section may be treated as an appendix, to be consulted at need.

The development proper starts with section 2. We introduce the main players and fix the basic notation. We take a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in the matrix algebra  $M_{p^r}(F)$ ,  $r \geq 1$ , as in part 4 above, and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  of endo-class  $\Theta$ . Thus  $\Theta \in \mathcal{E}^C(F)$  and  $\varsigma_\Theta = m/p^r$ . The Interpolation Theorem of [13] readily yields  $\Psi_\Theta(x) = p^{-r}\psi_{F[\alpha]/F}(x)$  in the range  $0 \leq x \leq \varsigma_\Theta/2$ . In the region  $\varsigma_\Theta/2 < \Psi_\Theta(x) \leq \varsigma_\Theta$ , it interprets the value  $\Psi_\Theta(x)$  in terms of intertwining properties of certain simple strata.

Section 3 is devoted to the proof of Theorem 1. The argument develops from [13] 8.3 and is couched almost entirely in terms of Galois representations. Take  $\sigma \in \widehat{W}_F^{\text{wr}}$  of dimension  $> 1$ . After a tame base field extension, 8.3 Theorem of [13] gives a sufficiently canonical presentation  $\sigma = \text{Ind}_{K/F} \tau$ , where  $K/F$  is cyclic of degree  $p$ . After an elementary change of variables, the jumps of  $\Sigma_\tau$  are among those of  $\Sigma_\sigma$  but one or two of them are “flattened”, in an obvious sense. One of these is invariably the first. If  $\sigma$  is of Carayol type, the other is the last. This follows from an application of the conductor formula of [14] 6.5 Theorem. That also gives a relation between the first and last jumps. One may then assume that  $\tau$  has the symmetry property and proceed by induction.

Section 4 makes a transition back to the GL-side. The combination of convexity and symmetry imposes significant restrictions on the piecewise linear graph  $y = \Psi_\Theta(x)$  in the relevant region  $0 \leq x \leq \varsigma_\Theta$ . We abstract these properties in the definition of the bi-Herbrand function  ${}^2\Psi_{(E/F, \varsigma)}$ . Much of the section is devoted to listing elementary, but useful, geometric properties of the graphs of  $\Psi_\Theta$  and  ${}^2\Psi_{(E/F, \varsigma)}$ . Our strategy is to identify  $\Psi_\Theta$  as a bi-Herbrand function. In many cases, one can do it straightaway, using only the easy consequences of the Interpolation Theorem in section 2 and the geometry of the graph: see 4.6 Example.

Sections 5 and 6 are highly technical in nature, preparing the way for the arguments of section 7. In section 5, we use the Interpolation Theorem to identify, via some delicate intertwining and conjugacy arguments, a subset of  $\|\mathcal{C}(\mathfrak{a}, \alpha)\|$

on which the Herbrand function  $\Psi_\Theta$  takes the expected value  ${}^2\Psi_{(F[\alpha]/F, \varsigma_\Theta)}$ . This is 5.3 Theorem, the main result of the section. The specification of this set, which we temporarily call  $\mathcal{L}_\alpha$ , can be quite subtle. There is nothing canonical or natural about  $\mathcal{L}_\alpha$ , but it is a very useful computational device.

The set  $\mathcal{C}(\mathfrak{a}, \alpha)$  does not determine  $\alpha$ , although it does determine  $\mathfrak{a}$  and the integer  $m$ . Let  $P(\mathfrak{a}, \alpha)$  be the set of  $\beta \in \mathrm{GL}_{p^r}(F)$  for which  $[\mathfrak{a}, m, 0, \beta]$  is a simple stratum satisfying  $\mathcal{C}(\mathfrak{a}, \beta) = \mathcal{C}(\mathfrak{a}, \alpha)$ . In section 6, we examine various ways in which one can construct elements of  $P(\mathfrak{a}, \alpha)$  while keeping track of consequent changes in the set  $\mathcal{L}_\alpha$ .

In section 7, we first define the subset  $\mathcal{C}^*(\mathfrak{a}, \alpha)$  of simple characters  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  that *conform to*  $\alpha$ . We show that, if  $\theta' \in \mathcal{C}(\mathfrak{a}, \alpha)$ , there exists  $\alpha'$  such that  $\mathcal{C}(\mathfrak{a}, \alpha') = \mathcal{C}(\mathfrak{a}, \alpha)$  and to which  $\theta'$  conforms. The calculations in sections 5 and 6 give a preliminary result (7.2 Theorem 1) from which Theorem 2 follows quickly.

With section 8, we return to the Galois side. We first re-cast the general theory of representations of, loosely speaking, Heisenberg type and so identify the class of representations with Herbrand function having a single jump. This is in preparation for section 9, where we prove Theorem 3. We conclude section 9 with an application of Theorems 2 and 3 to a question left open in section 8. Theorems 4A and 4B follow in section 10, where we combine and compare the main results of sections 7 and 9.

## BACKGROUND AND NOTATION

General notations are quite standard:  $\mathfrak{o}_F$  is the discrete valuation ring in  $F$ ,  $\mathfrak{p}_F$  is the maximal ideal of  $\mathfrak{o}_F$  and  $v_F$  is the normalized additive valuation. For  $k \geq 1$ ,  $U_F^k$  is the congruence unit group  $1 + \mathfrak{p}_F^k$ . Similarly, if  $\mathfrak{a}$  is a hereditary  $\mathfrak{o}_F$ -order in some matrix algebra, then  $U_{\mathfrak{a}}^k = 1 + \mathfrak{p}^k$ , where  $\mathfrak{p}$  is the Jacobson radical  $\mathrm{rad} \mathfrak{a}$  of  $\mathfrak{a}$ . For real  $x$ ,  $x \mapsto [x]$  is the greatest integer function.

If  $E/F$  is a finite field extension,  $\psi_{E/F}$ ,  $\varphi_{E/F}$  are the classical Herbrand functions discussed in section 1. If  $E/F$  is Galois and  $\Gamma = \mathrm{Gal}(E/F)$ , then  $\Gamma_a$ ,  $\Gamma^a$ ,  $a \geq 0$ , are the ramification subgroups of  $\Gamma$  in the lower, upper numbering conventions of [37].

The symbols  $\mathcal{W}_F$ ,  $\widehat{\mathcal{W}}_F$ ,  $\mathcal{P}_F$ ,  $\widehat{\mathcal{P}}_F$ ,  $\widehat{\mathrm{GL}}_F$ ,  $\mathcal{E}(F)$ ,  ${}^L\Theta$ ,  $[\sigma]_0^+$ ,  $\mathcal{R}_F(x)$ ,  $\mathcal{R}_F^+(x)$  all retain the meaning given them in the introduction.

Notation concerned with simple characters is all taken from [15] and [3]. For

the special cases considered here, full definitions are given in 2.1–3. The broader summary in [2] may be found helpful.

Certain special notations recur sporadically. Their definitions may be found as follows:  $\varsigma_\Theta$  (2.1),  $\varsigma_\sigma$  (2.2),  $\widehat{W}_F^{\text{wr}}$  (3.2),  $\widehat{W}_F^{\text{awr}}$  (3.2),  $\mathfrak{E}^C(F)$  (2.3),  $j_\infty(E|F)$  (1.5),  $w_{E/F}$  (1.6),  $\mathcal{C}^\star$  (7.1).

## 1. Classical Herbrand functions

Let  $E/F$  be a finite, separable field extension. As we go through the paper, we rely repeatedly on properties of the classical Herbrand function  $\psi_{E/F}$  and its inverse  $\varphi_{E/F}$ . For Galois extensions  $E/F$ , many of these are to be found in [37]. In the general case, we develop them from the outline in [18]. Beyond that, we need estimates of the *jumps of  $\psi_{E/F}$* , that is, the discontinuities of the derivative  $\psi'_{E/F}(x)$ ,  $x > 0$ . With only minor changes, the formalism applies equally well to inseparable extensions  $E/F$ : we indicate how this is done in 1.7.

We conclude the section with what seems to be a novel result on the structure of a broad class of totally ramified extensions. We do not need this until near the end of the paper but it fits well in the present context. The reader may wish to skip that, or even the entire section, referring back to it as needed.

**1.1.** Let  $E/F$  be a finite Galois extension. The Herbrand function  $\psi_{E/F}(x)$  is defined, for  $x \geq -1$ , in [37] IV §3 but we shall always assume  $x \geq 0$ . If  $K/F$  is a Galois extension contained in  $E$ , then  $\psi_{E/F} = \psi_{E/K} \circ \psi_{K/F}$ . If the finite separable extension  $E/F$  is not Galois, we follow [18]. Let  $E'/F$  be a finite Galois extension containing  $E$ . The function  $\psi_{E'/E}$  is positive and strictly increasing, so we may set

$$(1.1.1) \quad \psi_{E/F} = \psi_{E'/E}^{-1} \circ \psi_{E'/F}.$$

Because of the transitivity property for Galois extensions, this definition does not depend on the choice of  $E'/F$ . The relation

$$(1.1.2) \quad \psi_{E/F} = \psi_{E/K} \circ \psi_{K/F}$$

then holds for any tower  $F \subset K \subset E$  of finite separable extensions. In all cases,  $\varphi_{E/F}$  shall be the inverse function for  $\psi_{E/F}$ ,

$$(1.1.3) \quad \varphi_{E/F} \circ \psi_{E/F}(x) = x = \psi_{E/F} \circ \varphi_{E/F}(x), \quad x \geq 0.$$

**Lemma.**

- (1) *If  $K/F$  is finite and tamely ramified, then  $\psi_{K/F}(x) = ex$ , where  $e = e(K|F)$ .*
- (2) *If  $E/F$  is finite separable and  $K/F$  is finite and tamely ramified, with  $e(K|F) = e$ , then  $\psi_{EK/K}(x) = e(EK|E) \psi_{E/F}(x/e)$ . If  $E/F$  is totally wildly ramified, then  $\psi_{EK/K}(x) = e \psi_{E/F}(x/e)$ .*

*Proof.* Part (1) follows immediately from the definitions here and in [37]. By (1.1.2) and part (1),  $\psi_{EK/F}(x) = \psi_{EK/K} \circ \psi_{K/F}(x) = \psi_{EK/K}(ex)$ . On the other hand,  $\psi_{EK/F}(x) = \psi_{EK/E} \circ \psi_{E/F}(x) = e(EK|E) \psi_{E/F}(x)$ , whence part (2) follows.  $\square$

The lemma reduces most questions to the totally wildly ramified case.

**1.2.** We list some properties of the graph  $y = \psi_{E/F}(x)$ ,  $x \geq 0$ .

**Proposition.** *Let  $E/F$  be a finite separable extension and write  $e = e(E|F) = e_0 p^r$ , where  $e_0$  is an integer not divisible by  $p$ .*

- (1) *The function  $\psi_{E/F}$  is continuous, piece-wise linear, strictly increasing and convex.*
- (2) *If  $x$  is sufficiently large, then  $\psi_{E/F}(x) = ex$ .*
- (3) *There exists  $\epsilon > 0$  such that  $\psi_{E/F}(x) = e_0 x$ , for  $0 \leq x < \epsilon$ .*
- (4) *The derivative  $\psi'_{E/F}$  is continuous except at a finite number of points.*

*Proof.* All assertions are standard when  $E/F$  is Galois, and (2)–(4) then follow from (1.1.2) in general. In (1), the first two properties are clear while, by (3),  $\psi'_{E/F}(x) = e_0 \geq 1$  for  $x$  positive and sufficiently small. It is enough, therefore to show that  $\psi_{E/F}$  is convex. By 1.1 Lemma (2), we need only prove that  $\psi_{EK/K}$  is convex for some finite tame extension  $K/F$ . We choose  $K/F$  to be the maximal tame sub-extension of the normal closure  $E'/F$  of  $E/F$ . This reduces us to the case in which  $E'/F$  is totally wildly ramified. If  $E = F$ , there is nothing to prove, so assume otherwise. The proper subgroup  $\text{Gal}(E'/E)$  of the finite  $p$ -group  $\text{Gal}(E'/F)$  is contained in a normal subgroup of index  $p$ . That is, there is a Galois sub-extension  $F'/F$  of  $E/F$  of degree  $p$ . In the relation  $\psi_{E/F} = \psi_{E/F'} \circ \psi_{F'/F}$ , the function  $\psi_{F'/F}$  is convex since  $F'/F$  is Galois. By induction on degree,  $\psi_{E/F'}$  is convex, whence so is  $\psi_{E/F}$ .  $\square$

This technique of the proof of the proposition will be used again, so we make a formal definition.

**Definition.** Let  $E/F$  be a finite separable extension, with normal closure  $E'/F$ . Say that  $E/F$  is *absolutely wildly ramified* if  $E'/F$  is totally wildly ramified.

**1.3.** As in the Galois case, the function  $\psi_{E/F}$  reflects properties of the norm map  $N_{E/F} : E^\times \rightarrow F^\times$ .

**Proposition.** *Let  $E/F$  be a finite separable extension. Let  $\chi$  be a character of  $F^\times$  such that  $\text{sw}(\chi) = k \geq 1$ . The character  $\chi \circ N_{E/F}$  of  $E^\times$  then has the properties*

- (1)  $\text{sw}(\chi \circ N_{E/F}) \leq \psi_{E/F}(k)$  and,
- (2) if  $\psi'_{E/F}$  is continuous at  $k$ , then  $\text{sw}(\chi \circ N_{E/F}) = \psi_{E/F}(k)$ .

*Proof.* The result is standard when  $E/F$  is Galois [37] V Proposition 9. If  $E/F$  is tamely ramified, the function  $\psi_{E/F}$  is smooth (1.1 Lemma (1)), and the result follows from the Galois case.

Transitivity now reduces us to the case where  $E/F$  is totally wildly ramified. Also, if  $K/F$  is a finite tame extension, the result holds for  $E/F$  if and only if it holds for  $EK/K$ . We may therefore assume that  $E/F$  is absolutely wildly ramified. Let  $F'$  be a field,  $F \subset F' \subset E$ , such that  $F'/F$  is Galois of degree  $p$ . The result holds for the extension  $F'/F$  and so in general, by induction on  $[E:F]$ .  $\square$

**Definition.** A *jump* of  $\psi_{E/F}$  is a point  $x > 0$  at which the derivative  $\psi'_{E/F}$  is not continuous. Let  $J_{E/F}$  denote the set of jumps of  $\psi_{E/F}$ .

The set  $J_{E/F}$  is finite by 1.2 Proposition (4).

**Corollary.** *Let  $E/F$  be totally wildly ramified, and let  $K/F$  be a finite tame extension, with  $e = e(K|F)$ . If  $\chi$  is a character of  $K^\times$  with  $\text{sw}(\chi) = k \geq 1$ , such that  $e^{-1}k \notin J_{E/F}$ , then*

$$\text{sw}(\chi \circ N_{EK/K}) = \psi_{EK/F}(k) = e \psi_{E/F}(e^{-1}k).$$

*Proof.* The second equality is 1.1 Lemma. In that same vein,

$$e \psi_{E/F}(x) = \psi_{EK/E} \circ \psi_{E/F}(x) = \psi_{EK/F}(x) = \psi_{EK/K}(ex),$$

whence  $J_{EK/K} = e^{-1}J_{E/F}$ . The result now follows from the proposition.  $\square$

**1.4.** Another familiar property extends to the general case.

**Proposition.** *Let  $E/F$  be a finite separable extension. If  $\epsilon > 0$ , then*

$$\begin{aligned}\mathcal{R}_F(\epsilon) \cap \mathcal{W}_E &= \mathcal{R}_E(\psi_{E/F}(\epsilon)), \\ \mathcal{R}_F^+(\epsilon) \cap \mathcal{W}_E &= \mathcal{R}_E^+(\psi_{E/F}(\epsilon)).\end{aligned}$$

*Proof.* If  $E/F$  is Galois, the result follows from [37] IV Proposition 14. The case of  $E/F$  tame follows readily. If  $K/F$  is finite tame extension, the result therefore holds for  $E/F$  if and only if it holds for  $EK/K$  (cf. 1.1 Lemma). Thus we need only treat the case where  $E/F$  is absolutely wildly ramified. There is a Galois sub-extension  $F'/F$  of  $E/F$  of degree  $p$ . If  $F' = E$ , there is nothing to do, so we assume otherwise. We have

$$\begin{aligned}\mathcal{R}_F(\epsilon) \cap \mathcal{W}_E &= \mathcal{R}_F(\epsilon) \cap \mathcal{W}_{F'} \cap \mathcal{W}_E \\ &= \mathcal{R}_{F'}(\psi_{F'/F}(\epsilon)) \cap \mathcal{W}_E \\ &= \mathcal{R}_E(\psi_{E/F'}(\psi_{F'/F}(\epsilon))) = \mathcal{R}_E(\psi_{E/F}(\epsilon)),\end{aligned}$$

by induction on  $[E:F]$ . The second assertion follows.  $\square$

**1.5.** Let  $j_\infty(E|F)$  be the largest element of  $J_{E/F}$ .

**Proposition.** *Let  $E/F$  be separable and totally wildly ramified. If  $\bar{E}/F$  is the normal closure of  $E/F$ , then  $j_\infty(\bar{E}|F) = j_\infty(E|F)$ .*

*Proof.* The relation  $\psi_{\bar{E}/F} = \psi_{\bar{E}/E} \circ \psi_{E/F}$  implies that

$$J_{\bar{E}/F} = J_{E/F} \cup \psi_{E/F}^{-1}(J_{\bar{E}/E}).$$

We have to show that  $j_\infty(E|F)$  is the largest element of this set. Set  $\Gamma = \text{Gal}(\bar{E}/F)$  and  $\Delta = \text{Gal}(\bar{E}/E)$ . The definition of  $\Gamma_x$  [37] IV §1 gives  $\Delta_x = \Gamma_x \cap \Delta$ , for all  $x \geq 0$ . Let  $k_\infty$  be the largest jump of  $\Gamma$  in this numbering. Thus  $\Gamma_{k_\infty} \neq 1 = \Gamma_{k_\infty + \varepsilon}$ , for all  $\varepsilon > 0$ . As  $\bar{E}/F$  is the least Galois extension containing  $E$ , so  $\bigcap_{\gamma \in \Gamma} \gamma \Delta \gamma^{-1} = 1$ . That is,  $\Delta$  has no non-trivial subgroup normal in  $\Gamma$ . In particular,  $\Delta_{k_\infty} = \Gamma_{k_\infty} \cap \Delta = 1$ . The largest jump of  $\Delta$  is therefore strictly less than  $k_\infty$ . Translating back, the largest jump  $j_\infty(\bar{E}|E)$  of  $\psi_{\bar{E}/E}$  is strictly less than  $\psi_{E/F}(j_\infty(E|F))$ .  $\square$

**1.6.** Let  $E/F$  be a finite separable extension. Denote by  $d_{E/F}$  the differential exponent of  $E/F$ : thus  $\mathfrak{p}_E^{d_{E/F}}$  is the different of  $E/F$ . Define the *wild exponent*  $w_{E/F}$  of  $E/F$  by

$$(1.6.1) \quad w_{E/F} = d_{E/F} + 1 - e(E|F).$$

**Lemma.** *If  $F \subset K \subset E$  is a tower of finite separable extensions, then*

$$w_{E/F} = e(E|K)w_{K/F} + w_{E/K}.$$

*Proof.* The assertion follows from the multiplicativity property of the different and a short calculation.  $\square$

We develop this theme to get estimates relating the wild exponent  $w_{E/F}$  to the largest jump  $j_\infty(E|F)$  of  $\psi_{E/F}$ .

**Proposition.** *If  $E/F$  is separable and totally wildly ramified of degree  $p^r$ , then*

$$\psi_{E/F}(x) = p^r x - w_{E/F}, \quad x \geq j_\infty(E|F).$$

*Proof.* Let  $K/F$  be tamely ramified with  $e = e(K|F)$ . Thus  $w_{EK/K} = e w_{E/F}$  by the lemma. The result therefore holds for  $E/F$  if and only if it holds for  $EK/K$ . Taking  $K/F$  to be the maximal tame sub-extension of the normal closure of  $E/F$ , we reduce to the case where  $E/F$  is absolutely wildly ramified. Part (2) of 1.2 Proposition implies that there is a constant  $c_{E/F}$  such that  $\psi_{E/F}(x) = p^r x - c_{E/F}$ , for  $x \geq j_\infty(E|F)$ . We show that  $c_{E/F} = w_{E/F}$ .

Let  $F'/F$  be a sub-extension of  $E/F$  that is Galois of degree  $p$ . In this case,  $j_\infty(F'|F)$  is the only jump of  $\psi_{F'/F}$ , and it equals  $w_{F'/F}/(p-1)$  [37] V §3. The proposition thus holds for  $F'/F$ . If  $E/F$  is Galois, we may assume inductively that  $c_{E/F'} = w_{E/F'}$ . So, taking  $x$  sufficiently large, we get

$$\begin{aligned} p^r x - c_{E/F} &= \psi_{E/F'}(\psi_{F'/F}(x)) = \psi_{E/F'}(px - w_{F'/F}) \\ &= p^r x - p^{r-1} w_{F'/F} - w_{E/F'} = p^r x - w_{E/F}, \end{aligned}$$

by the lemma. Thus  $c_{E/F} = w_{E/F}$  when  $E/F$  is Galois.

Suppose that  $E/F$  is not Galois. The normal closure  $E'/F$  of  $E/F$  is totally wildly ramified by hypothesis. So, with  $p^s = [E':F]$  and  $x$  sufficiently large, we get

$$\begin{aligned} \psi_{E'/F}(x) &= p^s x - w_{E'/F} = \psi_{E'/E}(\psi_{E/F}(x)) \\ &= p^{s-r}(p^r x - c_{E/F}) - w_{E'/E}. \end{aligned}$$

Thus  $w_{E'/F} = e(E'|E)c_{E/F} - w_{E'/E}$ , and the lemma implies  $c_{E/F} = w_{E/F}$ .  $\square$

**Corollary.** *Let  $E/F$  be totally wildly ramified of degree  $p^r$ . If  $j_\infty = j_\infty(E|F)$  is the largest jump of  $\psi_{E/F}$ , then*

$$(p^r - 1)j_\infty \geq w_{E/F} \geq p^{r-1}(p-1)j_\infty \geq p^r j_\infty / 2.$$

Moreover,  $w_{E/F} = (p^r - 1)j_\infty$  if and only if  $j_\infty$  is the only jump of  $\psi_{E/F}$ .

*Proof.* Since  $\psi_{E/F}(x) \geq x$  for all  $x \geq 0$ , the first inequality follows directly from the proposition, likewise the final remark.

Observe that  $\psi'_{E/F}(x) \leq p^{r-1}$ , for all points  $0 < x < j_\infty$  at which the derivative is defined. The function  $\vartheta(x) = \psi_{E/F}(x) - p^{r-1}x$  is therefore decreasing on the interval  $0 < x < j_\infty$ . Thus  $\vartheta(j_\infty) \leq 0$ , or  $p^r j_\infty - w_{E/F} \leq p^{r-1} j_\infty$ , as required.  $\square$

**1.7.** If  $E/F$  is a finite, purely inseparable extension, we set  $\psi_{E/F}(x) = x$ ,  $x \geq 0$ . If  $E/F$  is a finite extension, define

$$(1.7.1) \quad \psi_{E/F} = \psi_{E/E_0} \circ \psi_{E_0/F} = \psi_{E_0/F},$$

where  $E_0/F$  is the maximal separable sub-extension of  $E/F$ . Assuming  $E \neq E_0$ , the derivative of  $\psi_{E/F}$  satisfies  $\psi'_{E/F}(x) < [E:F]$  for all  $x$ . We therefore set  $j_\infty(E|F) = \infty$  when  $E/F$  is not separable. With these definitions, all the results of 1.1–3, 1.5 and 1.6 remain valid.

**1.8.** We anticipate a phenomenon arising later on, in sections 5 and 6.

Let  $E/F$  be totally ramified of degree  $p^r$ ,  $r \geq 1$ . Thus  $E = F[\alpha]$ , where  $\alpha$  is a root of an Eisenstein polynomial  $f(X) = X^{p^r} + a_1 X^{p^r-1} + \cdots + a_{p^r-1} X + a_{p^r} \in \mathfrak{o}_F[X]$ , and one has  $d_{E/F} = v_E(f'(\alpha))$ .

Set  $a_0 = 1$ . If  $E/F$  is inseparable, the coefficient  $a_j$  is zero unless  $j \equiv 0 \pmod{p}$ . Each term  $(p^r - j)a_j \alpha^{j-1}$  in  $f'(\alpha)$  vanishes, giving  $d_{E/F} = w_{E/F} = \infty$ .

**Proposition.** *Suppose  $E/F$  is separable and totally ramified of degree  $p^r$ . There is an integer  $k$  such that  $0 \leq k \leq p^r - 1$ , and*

$$d_{E/F} = \min_{0 \leq j \leq p^r-1} v_E((p^r - j)a_j \alpha^{j-1}) \equiv k-1 \pmod{p^r}.$$

In particular,  $w_{E/F} \equiv k \pmod{p}$ . If  $F$  has characteristic  $p$ , then  $k \not\equiv 0 \pmod{p}$ .



*Proof.* For  $0 \leq j \leq p^r - 1$ , the term  $(p^r - j)a_j\alpha^{j-1}$  is either zero or

$$v_E((p^r - j)a_j\alpha^{j-1}) \equiv j - 1 \pmod{p^r}.$$

This gives the expression for  $d_{E/F}$ . If  $F$  has characteristic  $p$ , any term with  $j \equiv 0 \pmod{p}$  has valuation  $\infty$  and the second assertion follows.  $\square$

If  $F$  has characteristic zero, an Eisenstein polynomial  $f(X) = X^p - a$  gives a field extension  $E/F$  of degree  $p$  such that  $w_{E/F} \equiv 0 \pmod{p}$ .

**1.9.** We prove a simple, but under-appreciated, result concerning absolutely wildly ramified extensions  $E/F$  (1.2 Definition). It re-appears naturally in the analysis of representations in section 9.

Let  $E/F$  be a finite separable extension. As before, let  $J_{E/F}$  be the set of jumps of the piecewise linear function  $\psi_{E/F}$ . For  $x > 0$ , define

$$w_x(E|F) = \lim_{\epsilon \rightarrow 0} \psi'_{E/F}(x + \epsilon) / \psi'_{E/F}(x - \epsilon).$$

Thus  $w_x(E|F)$  is a non-negative power of  $p$ , while  $w_x(E|F) > 1$  if and only if  $x \in J_{E/F}$ .

If  $E/F$  is a finite Galois extension with  $\text{Gal}(E/F) = \Gamma$ , we use the notation  $\Gamma^{y+} = \bigcup_{z > y} \Gamma^z$ , and similarly for the lower numbering.

**Proposition.** *Let  $E/F$  be separable and absolutely wildly ramified, with normal closure  $L/F$ . Let  $a$  be the least element of  $J_{E/F}$ .*

- (1) *The number  $a$  is an integer and the least element of  $J_{L/F}$ .*
- (2) *Let  $T = T_1(E|F)$  be the group of characters  $\chi$  of  $F^\times$  such that  $\text{sw}(\chi) \leq a$  and  $\chi \circ N_{E/F} = 1$ . All non-trivial elements of  $T$  have Swan exponent  $a$ , and  $T$  is elementary abelian of order  $w_a(E|F)$ .*
- (3) *If  $E_1/F$  is class field to the group  $T$ , then  $F \subset E_1 \subset E$ ,  $\psi_{E_1/F}(a) = a$  and*

$$J_{E/E_1} = \psi_{E_1/F}(J_{E/F}) \setminus \{a\}.$$

*Proof.* Treat first the case where  $E/F$  is Galois. Set  $\Gamma = \text{Gal}(E/F)$  and write  $w_a(E|F) = p^s$ . Here we have  $\Gamma = \Gamma^a = \Gamma_a$ , and  $\Gamma^a / \Gamma^{a+}$  is elementary abelian of order  $p^s$ . Certainly  $a$  is an integer, being a jump of  $\varphi_{E/F} = \psi_{E/F}^{-1}$ . The number  $p^s = w_a(E|F)$  is the norm index  $(U_F^a : U_F^{1+a} N_{E/F}(U_E^a))$  ([37] V §6 Proposition 9), so there are exactly  $p^s$  characters  $\phi$  of  $U_F^a / U_F^{1+a}$  for which  $\phi \circ N_{E/F}$  is trivial.

On the other hand, if  $b < a$  then the norm map  $N_{E/F}$  induces an isomorphism  $U_E^1/U_E^{1+b} \rightarrow U_F^1/U_F^{1+b}$  *loc. cit.* It follows that there are exactly  $p^s$  characters  $\chi$  of  $F^\times$ , with  $\text{sw}(\chi) \leq a$ , such that  $\chi \circ N_{E/F} = 1$ . If  $\chi \neq 1$  has this property, then  $\text{sw}(\chi) = a$ . All assertions follow readily in this case.

We now proceed by induction on  $[E:F]$ . If  $[E:F] = p$  then, since  $E/F$  is absolutely wild, it is Galois and there is nothing to do. Assume, therefore, that  $[E:F] \geq p^2$ . Let  $L/F$  be the normal closure of  $E/F$  and write  $\Gamma = \text{Gal}(L/F)$ ,  $\Delta = \text{Gal}(L/E)$ . The transitivity formula  $\psi_{L/F} = \psi_{L/E} \circ \psi_{E/F}$  (1.1.2) implies  $J_{E/F} \subset J_{L/F}$ . So, if  $b$  is the least element of  $J_{L/F}$ , then  $b$  is an integer and  $b \leq a$ . We show that  $b = a$ .

Suppose, for a contradiction, that  $b < a$ . As  $b$  is not a jump of  $\psi_{E/F}$ , so  $b = \psi_{E/F}(b)$  lies in  $J_{L/E}$ . Likewise, if  $c \in J_{L/E}$ , then  $\varphi_{E/F}(c) \in J_{L/F}$ , so  $b$  is the least element of  $J_{L/E}$ . Let  $\phi$  be a character of  $E^\times$ , with  $\text{sw}(\phi) = b$ , such that  $\phi \circ N_{L/E} = 1$ . Since  $b < a$ , 1.3 Proposition implies that  $N_{E/F}$  induces an isomorphism  $U_E^1/U_E^{c+1} \rightarrow U_F^1/U_F^{c+1}$ , provided  $1 \leq c \leq b$ . It follows that there is a unique character  $\xi$  of  $F^\times$  satisfying  $\text{sw}(\xi) = b$  and  $\xi \circ N_{E/F} = \phi$ . Necessarily,  $\xi \circ N_{L/F} = 1$ , so we have our contradiction and this completes the proof of part (1).

In (2), as  $E/F$  is totally wildly ramified,  $T$  is an abelian  $p$ -group. Let  $\chi$  be a character of  $F^\times$ ,  $\chi \neq 1$ , and suppose that  $\text{sw}(\chi) = b < a$ . Since  $b \notin J_{L/F}$ ,  $\chi \circ N_{L/F}$  is not trivial. Thus  $\chi \circ N_{E/F} \neq 1$  and  $\chi \notin T$ . This proves the first assertion in (2). If  $\chi \in T$ ,  $\chi \neq 1$ , then  $\chi^p \in T$  and  $\text{sw}(\chi^p) < \text{sw}(\chi)$ . It follows that  $T$  is elementary abelian. Define  $E_1/F$  as in (3). The elements of  $T$ , viewed as characters of  $\Gamma = \Gamma_a = \Gamma^a$ , are trivial on  $\Delta$  and on the ramification subgroup  $\Gamma^{a+} = \Gamma_{a+}$ . Since  $\Gamma/\Gamma_{a+}$  is elementary abelian, the group  $\Phi = \Delta\Gamma_{a+}$  is normal in  $\Gamma$ , and equals  $\text{Gal}(L/E_1)$ . We next show that  $a = \psi_{E_1/F}(a)$  is not a jump of  $\psi_{E/E_1}$ . Suppose otherwise. Observe that  $\Phi_{a+} = \Gamma_{a+} \cap \Phi = \Gamma_{a+}$ . Let  $\xi$  be a character of  $E_1^\times$  of exponent  $a$ , such that  $\xi \circ N_{E/E_1} = 1$ . Thus  $\xi$  is trivial on  $\Delta$  and on  $\Gamma_{a+} = \Phi_{a+}$ , whence  $\xi = 1$ . By induction on degree, we deduce that  $a \notin J_{E/E_1}$ , and all assertions follow.  $\square$

Remark that

$$(1.9.1) \quad \psi_{E_1/F}(x) = \begin{cases} x, & 0 \leq x \leq a, \\ a + p^s(x-a), & a \leq x, \end{cases}$$

where  $p^s = [E_1:F]$ .

One may now apply the proposition to the absolutely wildly ramified extension  $E/E_1$  and iterate. This process leads directly to:

**Corollary 1.** *Let  $E/F$  be separable and absolutely wildly ramified. Let*

$$j_1 < j_2 < \cdots < j_t$$

*be the set of jumps of  $\psi_{E/F}$ . There is a unique tower of fields*

$$(1.9.2) \quad F = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_t = E$$

*with the following properties.*

- (1) *For  $1 \leq k \leq t$ , the extension  $E_k/E_{k-1}$  is elementary abelian of degree  $w_{j_k}(E|F)$ .*
- (2) *For  $1 \leq k \leq t$ , the function  $\psi_{E_k/E_{k-1}}$  has a unique jump, namely  $\psi_{E_{k-1}/F}(j_k)$ .*

We refer to the tower (1.9.2) as the *elementary resolution* of the absolutely wild extension  $E/F$ . It gives a factorization

$$(1.9.3) \quad \psi_{E/F} = \psi_{E_t/E_{t-1}} \circ \psi_{E_{t-1}/E_{t-2}} \circ \cdots \circ \psi_{E_2/E_1} \circ \psi_{E_1/F}$$

in which each factor  $\psi_{E_k/E_{k-1}}$ ,  $1 \leq k \leq t$ , has exactly one jump.

We finish with a more elementary consequence. This could have been done earlier but is much cheaper at this stage.

**Corollary 2.** *If  $E/F$  is totally wildly ramified, the values of  $\psi'_{E/F}$  are divisors of  $[E:F]$ .*

*Proof.* If  $E/F$  is absolutely wildly ramified, the result follows from (1.9.1) and (1.9.3). The general case is then given by 1.1 Lemma.  $\square$

## 2. Certain simple characters

The first part of this section, namely 2.1–3, provides a brief *aide mémoire* for those facts and methods from [3], [13] and [15] that will be used frequently. It relies on parts 2 and 3 of the Introduction for background but is focused more on the detail of the special cases with which we are concerned. The later subsections give partial results concerning Herbrand functions of endo-classes. The notation we set out here remains standard throughout the paper.

**2.1.** Let  $\mathfrak{E}(F)$  be the set of endo-classes of simple characters over  $F$ . When working with this set, we follow the scheme of [13] 4.2 (apart from one minor adjustment of notation).

To each  $\Theta \in \mathfrak{E}(F)$ , one attaches positive integer invariants  $\deg \Theta$ ,  $e_\Theta$  and a non-negative rational invariant  $\varsigma_\Theta$ . (In [13],  $\varsigma_\Theta$  is  $m_\Theta$ .) We will never be concerned with the case  $\varsigma_\Theta = 0$ , so assume  $\varsigma_\Theta > 0$ . Let  $\mu_F$  be a character of  $F$  of level one. By definition,  $\mu_F$  is trivial on  $\mathfrak{p}_F$ , but not trivial on  $\mathfrak{o}_F$ . There exists a simple stratum  $[\mathfrak{a}, m, 0, \beta]$  in a matrix ring  $M_n(F)$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, 0, \beta, \mu_F)$  of endo-class  $\Theta$ . (Here, we have used the full notation of [15] (3.2.1), (3.2.3), but we almost invariably abbreviate it to  $\mathcal{C}(\mathfrak{a}, \beta)$ .) The algebra  $E = F[\beta]$  is a field and

$$\deg \Theta = [E:F], \quad e_\Theta = e(E|F), \quad \varsigma_\Theta = m/e_\mathfrak{a},$$

where  $e_\mathfrak{a}$  is the  $\mathfrak{o}_F$ -period of the hereditary  $\mathfrak{o}_F$ -order  $\mathfrak{a}$ . We shall say that  $\theta$  is a *realization* of  $\Theta$  on  $[\mathfrak{a}, m, 0, \beta]$ , and that  $E/F$  is a *parameter field* for  $\Theta$ .

While  $\deg \Theta$ ,  $e_\Theta$  and  $\varsigma_\Theta$  are invariants of  $\Theta$ , there will often be many choices for the field extension  $E/F$ , even up to isomorphism. The number  $\varsigma_\Theta$  has a useful interpretation. If  $\pi \in \widehat{\mathrm{GL}}_F$  contains a simple character of endo-class  $\Theta$ , then, in the notation of the introduction,  $\varsigma_\Theta = \mathrm{sw}(\pi)/\mathrm{gr}(\pi)$ .

Let  $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ . Thus  $\sigma = {}^L\pi$ , for some  $\pi \in \widehat{\mathrm{GL}}_F$ . If  $\Theta$  is the endo-class of a simple character contained in  $\pi$ , then  $\mathrm{sw}(\sigma) = \mathrm{sw}(\pi)$  and

$$(2.1.1) \quad \mathrm{sw}(\sigma)/\dim \sigma = \mathrm{sw}(\pi)/\mathrm{gr}(\pi) = \varsigma_\Theta.$$

**2.2.** Attached to  $\Theta \in \mathfrak{E}(F)$  is a *structure function*  $\Phi_\Theta(x)$ ,  $x \geq 0$ . This is given by an explicit formula (4.4.1) of [13] which we do not need to repeat. If  $\pi \in \widehat{\mathrm{GL}}_F$  contains a simple character of endo-class  $\Theta$ , the definition gives

$$(2.2.1) \quad \Phi_\Theta(0) = \mathrm{sw}(\check{\pi} \times \pi)/\mathrm{gr}(\pi)^2.$$

Let  $\sigma \in \widehat{\mathcal{W}}_F$ . The orbit  $[\sigma]_0^+ \in \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$  and the canonical map  $\mathfrak{E}(F) \rightarrow \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$ ,  $\Theta \mapsto {}^L\Theta$ , are as in the Introduction.

Attached to  $\sigma$  is a *decomposition function*  $\Sigma_\sigma(x)$ ,  $x \geq 0$ , defined as follows [13] (3.1.2). Let  $\sigma$  act on the vector space  $V$ , so that the semisimple representation  $\check{\sigma} \otimes \sigma$  acts on  $X = \check{V} \otimes V$ . For  $\delta > 0$ , let  $X(\delta)$  be the space of  $\mathcal{R}_F^+(\delta)$ -fixed points in  $X$ . This has a unique  $\mathcal{R}_F^+(\delta)$ -complement  $X'(\delta)$ . The spaces  $X(\delta)$ ,  $X'(\delta)$  provide semisimple, smooth representations of  $\mathcal{W}_F$ . One sets

$$(2.2.2) \quad \Sigma_\sigma(\delta) = (\dim \sigma)^{-2}(\delta \dim X(\delta) + \mathrm{sw} X'(\delta)).$$

The function  $\Sigma_\sigma$  depends only on the orbit  $[\sigma]_0^+ \in \mathcal{W}_F \setminus \widehat{\mathcal{P}}_F$ .

Obviously,  $\Sigma_\sigma(0) = \text{sw}(\check{\sigma} \otimes \sigma) / (\dim \sigma)^2$ . Let  $\sigma = {}^L\pi$ ,  $\pi \in \widehat{\text{GL}}_F$ , and let  $\Theta$  be the endo-class of a simple character contained in  $\pi$ . Since the Langlands correspondence preserves Swan exponents of pairs, we have

$$\Sigma_\sigma(0) = \frac{\text{sw}(\check{\sigma} \otimes \sigma)}{(\dim \sigma)^2} = \frac{\text{sw}(\check{\pi} \times \pi)}{\text{gr}(\pi)^2} = \Phi_\Theta(0).$$

**Definition 1.** Let  $\Theta \in \mathcal{E}(F)$  and let  $\sigma \in \widehat{\mathcal{W}}_F$  satisfy  $[\sigma]_0^+ = {}^L\Theta$ . Define the *Herbrand function*  $\Psi_\Theta$  of  $\Theta$  by  $\Psi_\Theta = \Phi_\Theta^{-1} \circ \Sigma_\sigma$ .

The function  $\Psi_\Theta$  is continuous, strictly increasing and piecewise linear. It does not depend on the choice of  $\sigma$  in its definition. It satisfies  $\Psi_\Theta(0) = 0$  and  $\Psi_\Theta(x) = x$  for  $x \geq \varsigma_\Theta$ .

**Definition 2.** A *jump* of  $\Psi_\Theta$  is a point  $x$ ,  $0 < x < \varsigma_\Theta$ , at which  $\Psi'_\Theta$  is not continuous.

Except in trivial cases,  $\Psi'_\Theta$  has a discontinuity at  $\varsigma_\Theta$ , but that holds no interest and we exclude it as a jump. The derivative  $\Psi'_\Theta$  takes only finitely many values, and the function  $\Psi_\Theta$  has only finitely many jumps.

We often use the following property. Let  $K/F$  be a finite, tamely ramified field extension and set  $e = e(K|F)$ . Let  $\Theta^K \in \mathcal{E}(K)$  be a  $K/F$ -lift of  $\Theta$  [3] 9.7. By 7.1 Proposition of [13],

$$(2.2.3) \quad \Psi_\Theta(x) = \Psi_{\Theta^K}(ex)/e, \quad x \geq 0.$$

In Galois-theoretic terms, if  $\sigma \in \widehat{\mathcal{W}}_F$  and  $[\sigma]_0^+ = {}^L\Theta$ , then  ${}^L(\Theta^K) = [\tau]_0^+ \in \mathcal{W}_K \setminus \widehat{\mathcal{P}}_K$ , for some irreducible component  $\tau$  of  $\sigma|_{\mathcal{W}_K}$ : this follows from 6.2 Proposition of [12]. When  $\Theta$  is totally wild, it has a *unique*  $K/F$ -lift.

**2.3.** Let  $\Theta \in \mathcal{E}(F)$ . Say that  $\Theta$  is *totally wild* if  $\deg \Theta = e_\Theta = p^r$ , for an integer  $r \geq 0$ . So, if  $\Theta$  is totally wild and if  $E/F$  is a parameter field for  $\Theta$ , then  $E/F$  is totally ramified of degree  $p^r$ . If, in addition,  $r \geq 1$  and the integer  $m = p^r \varsigma_\Theta$  is not divisible by  $p$ , we say that  $\Theta$  is of *Carayol type* (cf. [17]).

**Notation.** Let  $\mathcal{E}^C(F)$  denote the set of  $\Theta \in \mathcal{E}(F)$  that are totally wild of Carayol type.

Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . There is a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M = M_{p^r}(F)$  carrying a realization of  $\Theta$ . We describe this, following the definitions

in Chapter 3 of [15]. The integer  $m$  is  $p^r \varsigma_\Theta$ , the field extension  $E = F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $m = -v_E(\alpha)$  is not divisible by  $p$ . The element  $\alpha$  is therefore *minimal over  $F$* , in the sense of [15] (1.4.14). We form the group

$$H^1(\alpha, \mathfrak{a}) = U_E^1 U_{\mathfrak{a}}^{1+[m/2]}.$$

Set  $\mu_M = \mu_F \circ \text{tr}_M$ , where  $\text{tr}_M : M \rightarrow F$  is the matrix trace. Define a function  $\mu_M * \alpha$  on  $M$  by

$$(2.3.1) \quad \mu_M * \alpha(x) = \mu_M(\alpha(x-1)), \quad x \in M.$$

In particular,  $\mu_M * \alpha$  represents a character of the group  $U_{\mathfrak{a}}^{1+[m/2]}$  which is trivial on  $U_{\mathfrak{a}}^{1+m}$ . The set  $\mathcal{C}(\mathfrak{a}, \alpha) = \mathcal{C}(\mathfrak{a}, 0, \alpha, \mu_M)$  consists of all characters  $\vartheta$  of  $H^1(\alpha, \mathfrak{a})$  such that  $\vartheta|_{U_{\mathfrak{a}}^{1+[m/2]}} = \mu_M * \alpha|_{U_{\mathfrak{a}}^{1+[m/2]}}$ . By hypothesis, there exists  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  of endo-class  $\Theta$ .

*Remarks.*

- (1) The endo-class of any  $\vartheta \in \mathcal{C}(\mathfrak{a}, \alpha)$  is totally wild of Carayol type.
- (2) Characters  $\vartheta_1, \vartheta_2 \in \mathcal{C}(\mathfrak{a}, \alpha)$  are endo-equivalent if and only if they are equal: this follows from [15] (3.3.2) and is peculiar to this situation.

**2.4.** We specialize to the case of  $\Theta \in \mathcal{E}^C(F)$ .

**Proposition.** *Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ .*

- (1) *The function  $\Phi_\Theta$  satisfies*

$$\Phi_\Theta(x) = \Phi_\Theta(0) + p^{-r}x, \quad 0 \leq x \leq \varsigma_\Theta.$$

- (2)  *$\Psi_\Theta(0) = 0$  and  $\Psi_\Theta(x) = x$ , for  $x \geq \varsigma_\Theta$ .*
- (3) *There exists  $\varepsilon > 0$  such that*

$$\Psi'_\Theta(x) = \begin{cases} p^{-r}, & 0 < x < \varepsilon, \\ p^r, & \varsigma_\Theta - \varepsilon < x < \varsigma_\Theta. \end{cases}$$

- (4) *The function  $\Psi_\Theta$  is convex in the region  $0 < x < \varsigma_\Theta$ .*
- (5) *If  $0 < x < \varsigma_\Theta$ , then  $0 < \Psi_\Theta(x) < x$ .*
- (6) *The jumps of  $\Psi_\Theta$  are the discontinuities of  $\Sigma'_\sigma(x)$ .*
- (7) *If  $\varsigma_\Theta = m/p^r$ , where  $m$  is not divisible by  $p$ , then*

$$(2.4.1) \quad \Phi_\Theta(0) = \Sigma_\sigma(0) = m(p^r - 1)/p^{2r}.$$

*Proof.* Part (1) is the definition [13] (4.4.1), and part (2) has already been noted. Part (3) is an instance of [13] 7.6 Proposition. The function  $\Sigma_\sigma$  is convex (2.2.2), and so (4) follows from (1). Part (5) now follows from (4) and (3). Part (6) follows from (1). Part (7) follows from (2.2.1) and [13] 4.1 Proposition.  $\square$

**2.5.** Key arguments will rely on the Interpolation Theorem of [13] 7.5. We give an overview of that result, as it applies to  $\Theta \in \mathcal{E}^C(F)$ .

**Definition.** A *twisting datum* over  $F$  is a triple  $(k, c, \chi)$  in which

- (1)  $k \geq 1$  is an integer;
- (2)  $c$  is an element of  $F$  such that  $v_F(c) = -k$ ;
- (3)  $\chi$  is a character of  $F^\times$ , of Swan exponent  $k$ , such that

$$\chi(x) = \mu_F * c(x), \quad x \in U_F^{1+[k/2]}.$$

Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . Suppose that  $\Theta$  is the endo-class of  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ , exactly as in 2.3. If  $(k, c, \chi)$  is a twisting datum over  $F$ , the character  $\chi \circ \det$  of  $\mathrm{GL}_{p^r}(F)$  satisfies

$$\chi(\det x) = \mu_M * c(x), \quad x \in U_{\mathfrak{a}}^{1+[p^r k/2]}.$$

Following the discussion in [13] 7.4, the quadruple  $[\mathfrak{a}, m, 0, \alpha+c]$  is a simple stratum in  $M$ , such that  $H^1(\alpha+c, \mathfrak{a}) = H^1(\alpha, \mathfrak{a})$ . The character  $\chi\theta : x \mapsto \chi(\det x)\theta(x)$ ,  $x \in H^1(\alpha, \mathfrak{a})$ , lies in  $\mathcal{C}(\mathfrak{a}, \alpha+c)$ . Denote by  $\chi\Theta$  the endo-class of  $\chi\theta$ .

Let  $\mathbb{A}$  be the ultrametric on  $\mathcal{E}(F)$  defined in [13] 5.1. We first give a preliminary version of the result, which follows from [13] 7.3 Proposition.

**Proposition 1.** *Let  $k \geq 1$  be an integer that is not a jump of  $\Psi_\Theta$ . If  $(k, c, \chi)$  is a twisting datum over  $F$ , then  $\Psi_\Theta(k) = \mathbb{A}(\chi\Theta, \Theta)$ . In particular,  $\mathbb{A}(\chi\Theta, \Theta)$  depends only on  $k$ , not  $c$  or  $\chi$ .*

*Notes.*

- (1) In the context of the proposition,  $\mathbb{A}(\chi\Theta, \Theta) = t/p^r$ , where  $t$  is the least integer such that the characters  $\theta \mid H^{1+t}(\alpha, \mathfrak{a})$ ,  $\chi\theta \mid H^{1+t}(\alpha, \mathfrak{a})$  intertwine in  $\mathrm{GL}_{p^r}(F)$ : that is the definition of  $\mathbb{A}$  in this case.
- (2) The characters  $\theta \mid H^{1+t}(\alpha, \mathfrak{a})$ ,  $\chi\theta \mid H^{1+t}(\alpha, \mathfrak{a})$  intertwine in  $\mathrm{GL}_{p^r}(F)$  if and only if they are conjugate in  $\mathrm{GL}_{p^r}(F)$  [15] (3.5.11).
- (3) When  $k$  is a jump of  $\Psi_\Theta$ ,  $\mathbb{A}(\chi\Theta, \Theta)$  may depend on  $\chi$ , not only  $k$ .

We recall explicitly the notion of tame lifting, as it applies to  $\Theta \in \mathcal{E}^C(F)$ . Let  $K/F$  be a finite, tamely ramified field extension with  $e(K|F) = e$ . We form simple characters over  $K$  relative to the character  $\mu_K = \mu_F \circ \text{Tr}_{K/F}$  of  $K$ . There is a unique simple stratum in  $M_{p^r}(K)$  of the form  $[\mathfrak{a}^K, em, 0, \alpha]$ . Setting  $EK = K[\alpha] \subset M_{p^r}(K)$ , there is a unique  $\theta^K \in \mathcal{C}(\mathfrak{a}^K, \alpha)$  such that  $\theta^K(x) = \theta(N_{EK/E}(x))$ ,  $x \in U_{EK}^1$ . The endo-class  $\Theta^K$  of  $\theta^K$  lies in  $\mathcal{E}^C(K)$  and is the unique  $K/F$ -lift of  $\Theta$ . Combining Proposition 1 with (2.3.3), we get:

**Proposition 2.** *Let  $K/F$  be a finite tame extension with  $e = e(K|F)$ , and let  $\mathbb{A}_K$  be the canonical ultrametric on  $\mathcal{E}(K)$ . Let  $k \geq 1$  be an integer such that  $k/e$  is not a jump of  $\Psi_\Theta$ . If  $(k, c, \chi)$  is a twisting datum over  $K$ , then*

$$\Psi_\Theta(k/e) = \Psi_{\Theta^K}(k)/e = \mathbb{A}_K(\chi\Theta^K, \Theta^K)/e.$$

Proposition 2 summarizes the Interpolation Theorem.

**2.6.** Again let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$ . We choose a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M = M_{p^r}(F)$  carrying a realization  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  of  $\Theta$  (as in 2.3). In the remainder of the section, we use the Interpolation Theorem to determine  $\Psi_\Theta$  on part of its range.

**Proposition.** *Writing  $E = F[\alpha]/F$ , the Herbrand function  $\Psi_\Theta$  satisfies*

$$\Psi_\Theta(x) = p^{-r} \psi_{E/F}(x), \quad 0 \leq x \leq \varsigma_\Theta/2.$$

*Proof.* Let  $k$  be an integer,  $0 < k < \varsigma_\Theta/2$ , which is not a jump of either function  $\psi_{E/F}$ ,  $\Psi_\Theta$ . Let  $(k, \chi, c)$  be a twisting datum over  $F$ . The character  $\chi \circ \det$  of  $\text{GL}_{p^r}(F)$  is trivial on  $U_{\mathfrak{a}}^{1+p^r k}$ . Since  $p^r k \leq [m/2]$ , it is also trivial on the group  $U_{\mathfrak{a}}^{1+[m/2]}$ . The character  $\chi\theta : y \mapsto \chi(\det y)\theta(y)$ ,  $y \in H^1(\alpha, \mathfrak{a})$ , thus lies in  $\mathcal{C}(\mathfrak{a}, \alpha)$  (2.3). It follows from [15] (3.3.2) that the characters  $\chi\theta$ ,  $\theta$  intertwine on a group  $H^{1+t}(\alpha, \mathfrak{a}) = H^1(\alpha, \mathfrak{a}) \cap U_{\mathfrak{a}}^{1+t}$ ,  $t \geq 0$ , if and only if they are equal there.

On the other hand, by 2.5 Proposition 1,  $\Psi_\Theta(k) = t/p^r$ , where  $t$  is the least non-negative integer for which  $\theta$  and  $\chi\theta$  intertwine on  $H^{1+t}(\alpha, \mathfrak{a})$ . In this case,  $t$  is the least integer such that  $\chi \circ \det$  is trivial on  $U_E^{1+t}$ . We have  $\chi \circ \det(y) = \chi \circ N_{E/F}(y)$ ,  $y \in E^\times$ . That  $k$  is not a jump of  $\psi_{E/F}$  implies  $t = \psi_{E/F}(k)$  (1.3 Proposition).

In general, it is enough to prove the desired identity on a dense set of points  $x$ . Take  $x = a/b$ , for positive integers  $a$  and  $b$  with  $b$  not divisible by  $p$ . Assume



that  $x$  is not a jump of  $\psi_{E/F}$  or  $\Psi_\Theta$ . Let  $K/F$  be a finite, tamely ramified field extension with  $e(K|F) = b$ . If  $\Theta^K$  is the unique  $K/F$ -lift of  $\Theta$ , then  $bx$  is not a jump of  $\psi_{EK/K}$  or  $\Psi^{\Theta^K}$ . The first case of the argument, 2.5 Proposition 2 and 1.1 Lemma together yield

$$\Psi_\Theta(x) = \Psi_{\Theta^K}(a)/b = p^{-r}\psi_{EK/K}(a)/b = p^{-r}\psi_{E/F}(x),$$

as required.  $\square$

**2.7.** Remaining in the situation of 2.6, we refine the other part of 2.4 Proposition (3). We use the concept of *formal intertwining* of strata (as in [15] 2.6).

**Proposition.** *Let  $k$  be an integer,  $0 < k < \varsigma_\Theta$ , which is not a jump of  $\Psi_\Theta$ . Let  $t = p^r \Psi_\Theta(k)$ . If  $2t > m$ , then  $t$  is the least integer such that the strata  $[\mathfrak{a}, m, t, \alpha]$ ,  $[\mathfrak{a}, m, t, \alpha+c]$  intertwine formally.*

*Proof.* Let  $l$  be an integer such that  $2l > m$ . We have

$$\begin{aligned} \theta(x) &= \mu_M * \alpha(x), \\ \chi\theta(x) &= \mu_M * (\alpha+c)(x), \end{aligned} \quad x \in H^{1+l}(\alpha, \mathfrak{a}) = U_{\mathfrak{a}}^{1+l}.$$

In this situation, an element  $g$  of  $\mathrm{GL}_{p^r}(F)$  intertwines  $\theta|U_{\mathfrak{a}}^{1+l}$  with  $\chi\theta|U_{\mathfrak{a}}^{1+l}$  if and only if  $g^{-1}(\alpha + \mathfrak{p}^{-l})g \cap (\alpha+c+\mathfrak{p}^{-l}) \neq \emptyset$ , that is,  $g$  intertwines the strata  $[\mathfrak{a}, m, l, \alpha]$ ,  $[\mathfrak{a}, m, l, \alpha+c]$  formally. The result so follows from 2.5 Proposition 1.  $\square$

### 3. Functional equation

Let  $\Theta \in \mathcal{E}^C(F)$  (2.3 Notation) be of degree  $p^r$ . In particular,  $r \geq 1$ . In this section, we uncover a profound and surprising property of the function  $\Psi_\Theta$ .

**3.1.** The main result is:

**Theorem.** *Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$ ,  $r \geq 1$ . The Herbrand function  $\Psi_\Theta$  satisfies*

$$(3.1.1) \quad \varsigma_\Theta - x = \Psi_\Theta(\varsigma_\Theta - \Psi_\Theta(x)), \quad 0 \leq x \leq \varsigma_\Theta.$$

For many arguments, it is convenient to have an alternative formulation of (3.1.1).

**Symmetry.** *The function  $\Psi_\Theta$  satisfies  $0 \leq \Psi_\Theta(x) \leq x$ , for  $0 \leq x \leq \varsigma_\Theta$ . In that range, the graph  $y = \Psi_\Theta(x)$  is symmetric with respect to the line  $x+y = \varsigma_\Theta$ .*

The first assertion here is 2.4 Proposition (5). Reflection in the line  $x+y = \varsigma_\Theta$  is the map

$$i_{\varsigma_\Theta} : (x, y) \mapsto (\varsigma_\Theta - y, \varsigma_\Theta - x),$$

so the two formulations are indeed equivalent.

Before embarking on the proof of (3.1.1), we observe that it has a converse. As recalled in 2.5, the set  $\mathfrak{E}(F)$  carries a canonical action  $(\chi, \Theta) \mapsto \chi\Theta$  of the group of characters  $\chi$  of  $U_F^1$ . It has the property  $\Psi_{\chi\Theta} = \Psi_\Theta$  [13] 7.4 Proposition.

**Corollary.** *Let  $\Theta \in \mathfrak{E}(F)$  be totally wild, with  $\deg \Theta = p^r$ ,  $r \geq 1$ . Suppose that  $\varsigma_\Theta \leq \varsigma_{\chi\Theta}$ , for all characters  $\chi$  of  $U_F^1$ . The function  $\Psi_\Theta$  satisfies (3.1.1) if and only if  $\Theta \in \mathfrak{E}^C(F)$ .*

*Proof.* The hypothesis on  $\Theta$  is equivalent to  $\varsigma_\Theta = ap^{t-r}$ , for integers  $a \not\equiv 0 \pmod{p}$ ,  $0 \leq t < r$  (7.6 Remark of [13]). In particular,  $\Theta \in \mathfrak{E}^C(F)$  if and only if  $t = 0$ . By 7.6 Proposition of [13], there exist  $\epsilon > 0$ ,  $\delta > 0$ , such that

$$\Psi'_\Theta(x) = \begin{cases} p^{-r}, & 0 < x < \epsilon, \\ p^{r-t}, & \varsigma_\Theta - \delta < x < \varsigma_\Theta. \end{cases}$$

If the functional equation holds for  $\Theta$ , then  $t = 0$  and so  $\Theta \in \mathfrak{E}^C(F)$ . The converse is the theorem.  $\square$

The proof of (3.1.1) occupies the entire section. The first intermediate result, 3.4 Theorem, is entirely Galois-theoretic and applies to a relatively wide class of representations. The second, 3.5 Theorem, applies only to representations of Carayol type, and its proof depends on an intervention from the GL-side, in the form of a case of the conductor formula of [14]. That result forms the first step in an inductive proof of the theorem.

**3.2.** Let  $\sigma \in \widehat{\mathcal{W}}_F$ . Let  $\varsigma_\sigma$  be the *slope* of  $\sigma$ . That is,

$$(3.2.1) \quad \begin{aligned} \varsigma_\sigma &= \inf \{ \epsilon > 0 : \mathcal{R}_F(\epsilon) \subset \text{Ker } \sigma \} \\ &= \text{sw}(\sigma) / \dim \sigma, \end{aligned}$$

by [21] Théorème 3.5. If  $\varsigma_\sigma > 0$ , then  $\sigma|_{\mathcal{R}_F(\varsigma_\sigma)}$  does not contain the trivial character.

**Definition.**

- (1) Say that  $\sigma$  is *totally wild* if the restriction  $\sigma|_{\mathcal{P}_F}$  of  $\sigma$  to  $\mathcal{P}_F$  is irreducible. Let  $\widehat{\mathcal{W}}_F^{\text{wr}}$  be the set of totally wild elements  $\sigma$  of  $\widehat{\mathcal{W}}_F$ . In particular, if  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  then  $\dim \sigma = p^r$ , for an integer  $r \geq 0$ . Say that  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  is of *Carayol type* if  $p$  does not divide  $\text{sw}(\sigma)$  and  $\dim \sigma \neq 1$ .
- (2) Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  have dimension  $p^r$ . Say that  $\sigma$  is *absolutely wild* if the associated projective representation  $\bar{\sigma} : \mathcal{W}_F \rightarrow \text{PGL}_{p^r}(\mathbb{C})$  factors through a finite Galois group  $\text{Gal}(E/F)$ , with  $E/F$  totally wildly ramified. Write  $\widehat{\mathcal{W}}_F^{\text{awr}}$  for the set of absolutely wild elements  $\sigma$  of  $\widehat{\mathcal{W}}_F^{\text{wr}}$ .

**Lemma.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ . Let  $K/F$  be a finite, tamely ramified field extension and set  $e(K|F) = e$ . The representation  $\sigma^K = \sigma|_{\mathcal{W}_K}$  is irreducible. It lies in  $\widehat{\mathcal{W}}_K^{\text{wr}}$  and

$$\Sigma_\sigma(x) = e^{-1} \Sigma_{\sigma^K}(ex), \quad x \geq 0.$$

One may choose  $K/F$  so that  $\sigma^K \in \widehat{\mathcal{W}}_K^{\text{awr}}$ .

*Proof.* The relation between decomposition functions is [13] 3.2 Proposition. The projective representation  $\bar{\sigma}$  factors through a finite Galois group  $\text{Gal}(E/F)$ . The second assertion holds when  $K/F$  is the maximal tame sub-extension of  $E/F$ .  $\square$

**3.3.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ . Directly from the definition recalled in (2.2.2),  $\Sigma_\sigma(x) = x$ , for  $x > \varsigma_\sigma - \epsilon$  and some  $\epsilon > 0$ . Thus all discontinuities of  $\Sigma'_\sigma(x)$  lie in the region  $0 < x < \varsigma_\sigma$ . We call such points the *jumps* of  $\Sigma_\sigma$ .

We assemble some properties of absolutely wild representations.

**Lemma 1.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  have dimension  $p^r$ ,  $r \geq 1$ . Let  $a$  be the least jump of the function  $\Sigma_\sigma$ .

- (1) The jump  $a$  is an integer and there exists a character  $\chi$  of  $\mathcal{W}_F$ , with  $\text{sw}(\chi) = a$ , such that  $\chi \otimes \sigma \cong \sigma$ .
- (2) If  $\chi'$  is a non-trivial character of  $\mathcal{W}_F$  such that  $\chi' \otimes \sigma \cong \sigma$ , then  $\text{sw}(\chi') \geq a$ .
- (3) The character  $\chi$  of (1) has order  $p$ . If  $K/F$  is the cyclic extension such that  $\mathcal{W}_K = \text{Ker } \chi$ , there exists  $\tau \in \widehat{\mathcal{W}}_K^{\text{awr}}$  such that  $\sigma \cong \text{Ind}_{K/F} \tau$ . The representation  $\tau$  is uniquely determined up to conjugation by  $\text{Gal}(K/F)$ .
- (4) Suppose, in (3), that  $r \geq 2$ . The representation  $\tau$  is then of Carayol type if and only if  $\sigma$  is of Carayol type.

*Proof.* Parts (1)–(3) are [13] 8.3 Theorem. Let  $w_{K/F}$  be the wild exponent of the extension  $K/F$  (1.6.1). The standard formula  $\text{sw}(\sigma) = \text{sw}(\tau) + \dim(\tau) w_{K/F}$  gives  $\text{sw}(\sigma) \equiv \text{sw}(\tau) \pmod{p}$  and part (4) follows.  $\square$

Continuing in the situation of Lemma 1 we gather some standard facts from section 1 and [37], for convenience of reference.

**Lemma 2.**

- (1) *The point  $a$  is the unique ramification jump of the extension  $K/F$ , in either upper or lower numbering.*
- (2) *The group  $\mathcal{W}_K \cap \mathcal{R}_F(a)$  is of index  $p$  in  $\mathcal{R}_F(a)$  and  $\mathcal{R}_F^+(a) \subset \mathcal{W}_K$ , while  $\mathcal{W}_F = \mathcal{W}_K \mathcal{R}_F(a)$ .*
- (3) *The following relations hold:*

$$\begin{aligned} \mathcal{R}_K(\epsilon) &= \begin{cases} \mathcal{R}_F(\epsilon) \cap \mathcal{W}_K, & 0 < \epsilon \leq a, \\ \mathcal{R}_F(\varphi_{K/F}(\epsilon)), & a < \epsilon. \end{cases} \\ \mathcal{R}_K^+(\epsilon) &= \mathcal{R}_F^+(\varphi_{K/F}(\epsilon)), \quad a \leq \epsilon. \end{aligned}$$

- (4) *The Herbrand function  $\varphi_{K/F}$  is given by*

$$\varphi_{K/F}(x) = \begin{cases} x, & 0 \leq x \leq a, \\ a + (x-a)/p, & a \leq x. \end{cases}$$

**3.4.** As the first part of the proof of (3.1.1), we develop 3.3 Lemma 1 using the same notation. Thus  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  has dimension  $p^r$ ,  $r \geq 1$ . The first jump of  $\Sigma_\sigma$  is at  $a$ ,  $\chi$  is a character of  $\mathcal{W}_F$  such that  $\text{sw}(\chi) = a$  and  $\chi \otimes \sigma \cong \sigma$ . Again,  $\mathcal{W}_K = \text{Ker } \chi$  and  $\sigma = \text{Ind}_{K/F} \tau$ ,  $\tau \in \widehat{\mathcal{W}}_K^{\text{awr}}$ .

For  $\epsilon > 0$ , set

$$\begin{aligned} d_\epsilon(\sigma) &= \dim \text{End}_{\mathcal{R}_F(\epsilon)}(\sigma), \\ d_\epsilon^+(\sigma) &= \dim \text{End}_{\mathcal{R}_F^+(\epsilon)}(\sigma). \end{aligned}$$

Since  $\mathcal{R}_F(\epsilon)$ ,  $\mathcal{R}_F^+(\epsilon)$  are normal subgroups of the pro- $p$  group  $\mathcal{P}_F$ , the integers  $d_\epsilon(\sigma)$ ,  $d_\epsilon^+(\sigma)$  are non-negative powers of  $p$ . Referring back to the definition (2.2.2) of  $\Sigma_\sigma$ ,  $p^{-2r} d_\epsilon(\sigma)$  is the *left* derivative of the piecewise linear function  $\Sigma_\sigma$  at the point  $\epsilon$ . Likewise,  $p^{-2r} d_\epsilon^+(\sigma)$  is the *right* derivative of  $\Sigma_\sigma$  at  $\epsilon$ . It

follows that  $d_\epsilon(\sigma) = d_\epsilon^+(\sigma)$  unless  $\epsilon$  is a jump of  $\Sigma_\sigma$ . If  $\epsilon$  is a jump of  $\Sigma_\sigma$ , then  $w_\epsilon(\sigma) = d_\epsilon^+(\sigma)/d_\epsilon(\sigma)$  is a positive power of  $p$ . We have

$$\Sigma'_\sigma(x) = \begin{cases} p^{-2r}, & 0 < x < \varepsilon, \\ 1, & \varsigma_\sigma - \varepsilon < x, \end{cases}$$

for some  $\varepsilon > 0$ . Therefore

$$\prod_{\epsilon > 0} w_\epsilon(\sigma) = \prod_{\epsilon > 0} d_\epsilon^+(\sigma)/d_\epsilon(\sigma) = p^{2r}.$$

We make parallel definitions,

$$\begin{aligned} {}_F d_\epsilon(\tau) &= \dim \operatorname{End}_{\mathcal{R}_F(\epsilon) \cap \mathcal{W}_K}(\tau), \\ {}_F d_\epsilon^+(\tau) &= \dim \operatorname{End}_{\mathcal{R}_F^+(\epsilon) \cap \mathcal{W}_K}(\tau), \\ {}_F w_\epsilon(\tau) &= {}_F d_\epsilon^+(\tau)/{}_F d_\epsilon(\tau). \end{aligned}$$

The quotient  $w_\epsilon(\sigma)/{}_F w_\epsilon(\tau)$  is a power of  $p$ , and

$$(3.4.1) \quad \prod_{\epsilon > 0} w_\epsilon(\sigma)/{}_F w_\epsilon(\tau) = p^2.$$

*Remark.* One can define  $d_\epsilon(\tau)$ , etc., exactly as before, relative to the base field  $K$ . One then has  ${}_F d_\epsilon(\tau) = d_{\psi_{K/F}(\epsilon)}(\tau)$  (cf. 3.3 Lemma 2) and similarly for the other functions. We use this notation to simplify comparison between the two base fields  $F$  and  $K$ .

Using the notation from the start of the sub-section, we prove:

**Theorem.** *Let  $\gamma \in \operatorname{Gal}(K/F)$ ,  $\gamma \neq 1$ . The quantity*

$$(3.4.2) \quad c = c_{K/F}(\sigma) = \inf \{ \epsilon > 0 : \operatorname{Hom}_{\mathcal{R}_F(\epsilon) \cap \mathcal{W}_K}(\tau, \tau^\gamma) \neq 0 \}$$

*is independent of the choice of  $\gamma$ . The following properties hold.*

- (1)  $c \geq a$ .
- (2) *If  $c > a$ , then  $w_a(\sigma)/{}_F w_a(\tau) = w_c(\sigma)/{}_F w_c(\tau) = p$ , while  $w_\epsilon(\sigma)/{}_F w_\epsilon(\tau) = 1$  for all other values of  $\epsilon > 0$ .*
- (3) *If  $c = a$ , then  $w_a(\sigma)/{}_F w_a(\tau) = p^2$ , while  $w_\epsilon(\sigma)/{}_F w_\epsilon(\tau) = 1$  for all other values of  $\epsilon > 0$ .*

*Proof.* Let  $\epsilon > 0$ . The irreducible components of the semisimple representation  $\tau|_{\mathcal{W}_K \cap \mathcal{R}_F(\epsilon)}$  are all  $\mathcal{W}_K$ -conjugate and occur with the same multiplicity. Likewise for  $\tau^\gamma|_{\mathcal{W}_K \cap \mathcal{R}_F(\epsilon)}$ . Consequently,

$$\mathrm{Hom}_{\mathcal{R}_F(\epsilon) \cap \mathcal{W}_K}(\tau, \tau^\gamma) \neq 0 \iff \tau^\gamma|_{\mathcal{R}_F(\epsilon) \cap \mathcal{W}_K} \cong \tau|_{\mathcal{R}_F(\epsilon) \cap \mathcal{W}_K}.$$

This condition is surely independent of  $\gamma \neq 1$ . If  $0 < \epsilon < a$ , the function  $\Sigma_\sigma$  is smooth at  $\epsilon$ , whence  $\sigma|_{\mathcal{R}_F(\epsilon)}$  is irreducible. It is induced from  $\tau|_{\mathcal{W}_K \cap \mathcal{R}_F(\epsilon)}$  whence follows part (1) of the theorem.

To proceed, we need another litany of notation. Let  $\epsilon > 0$  and choose an irreducible component  $\sigma_\epsilon$  of  $\sigma|_{\mathcal{R}_F(\epsilon)}$ . Let  $l_\epsilon(\sigma)$  be the number of distinct  $\mathcal{W}_F$ -conjugates of  $\sigma_\epsilon$ , and  $m_\epsilon(\sigma)$  the multiplicity of  $\sigma_\epsilon$  in  $\sigma|_{\mathcal{R}_F(\epsilon)}$ . Thus  $d_\epsilon(\sigma) = l_\epsilon(\sigma)m_\epsilon(\sigma)^2$  while  $l_\epsilon(\sigma)m_\epsilon(\sigma)$  is the Jordan-Hölder length of  $\sigma|_{\mathcal{R}_F(\epsilon)}$ . All of these numbers are non-negative powers of  $p$ .

Similarly, choose an irreducible component  $\sigma_\epsilon^+$  of  $\sigma|_{\mathcal{R}_F^+(\epsilon)}$  and define  $l_\epsilon^+(\sigma)$ ,  $m_\epsilon^+(\sigma)$  in the same manner. Thus  $d_\epsilon^+(\sigma) = l_\epsilon^+(\sigma)m_\epsilon^+(\sigma)^2$  and  $l_\epsilon^+(\sigma)m_\epsilon^+(\sigma)$  is the Jordan-Hölder length of  $\sigma|_{\mathcal{R}_F^+(\epsilon)}$ , all being non-negative powers of  $p$ .

Likewise on replacing  $\sigma$  by  $\tau$ ,  $\mathcal{R}_F(\epsilon)$  by  $\mathcal{W}_K \cap \mathcal{R}_F(\epsilon)$  and  $\mathcal{R}_F^+(\epsilon)$  by  $\mathcal{W}_K \cap \mathcal{R}_F^+(\epsilon)$ . Note that, if  $\epsilon > a$ , we may take  $\sigma_\epsilon = \tau_\epsilon$  and  $\sigma_\epsilon^+ = \tau_\epsilon^+$  (cf. 3.3 Lemma 2). We may also take  $\sigma_a^+ = \tau_a^+$ .

**Lemma 1.** *If  $\Sigma_\sigma$  is smooth at a point  $\epsilon > 0$  then  $\Sigma_\tau$  is smooth at  $\psi_{K/F}(\epsilon)$ .*

*Proof.* Suppose first that  $\epsilon < a$ , so that  $\psi_{K/F}(\epsilon) = \epsilon$ . The definition of  $a$  ensures that the function  $\Sigma_\sigma$  is smooth at  $\epsilon$ . The representation  $\tau$  is irreducible on  $\mathcal{R}_K(a) = \mathcal{R}_F(a) \cap \mathcal{W}_K$ , and so also on  $\mathcal{R}_K(\epsilon)$ . It follows that  $\Sigma_\tau$  is smooth at  $\epsilon$ .

The function  $\Sigma_\sigma$  is not smooth at  $a$ , so take  $\epsilon > a$ . Since  $\Sigma_\sigma$  is smooth at  $\epsilon$ , 8.1 Proposition of [13] shows that the representation  $\sigma_\epsilon$  is irreducible on  $\mathcal{R}_F^+(\epsilon)$  and that  $\sigma_\epsilon$  is not  $\mathcal{W}_F$ -conjugate to  $\chi \otimes \sigma_\epsilon$ , for any character  $\chi \neq 1$  of  $\mathcal{R}_F(\epsilon)/\mathcal{R}_F^+(\epsilon)$ . Since we may take  $\tau_\epsilon = \sigma_\epsilon$ , the argument reverses to show that  $\Sigma_\tau$  is smooth at  $\psi_{K/F}(\epsilon)$ .  $\square$

We assume henceforth that  $\epsilon > a$  and use the notation introduced for Lemma 1. The  $\mathcal{W}_F$ -stabilizer of (the isomorphism class of)  $\sigma_\epsilon$  is of the form  $G_\epsilon = \mathcal{W}_{E_\epsilon}$ , for a finite field extension  $E_\epsilon/F$ . Likewise, let  $G_\epsilon^+ = \mathcal{W}_{E_\epsilon^+}$  denote the  $\mathcal{W}_F$ -stabilizer of  $\sigma_\epsilon^+$ . The  $\mathcal{W}_K$ -stabilizer of  $\tau_\epsilon = \sigma_\epsilon$  is then  $\mathcal{W}_K \cap G_\epsilon = \mathcal{W}_{KE_\epsilon}$ , and similarly with  $+$ 's.

**Lemma 2.** *If  $\epsilon > a$ , then*

$$\frac{d_\epsilon^+(\sigma)}{d_\epsilon(\sigma)} = \frac{{}_F d_\epsilon^+(\tau)}{{}_F d_\epsilon(\tau)} \frac{[K \cap E_\epsilon : F]}{[K \cap E_\epsilon^+ : F]}.$$

*The quotient of field degrees takes only the values 1 and  $p$ .*

*Proof.* Since  $\epsilon > a$ ,

$$\begin{aligned} m_\epsilon(\sigma) &= \sum_{\gamma \in \text{Gal}(K/F)} \dim \text{Hom}_{\mathcal{R}_F(\epsilon)}(\sigma_\epsilon, \tau^\gamma) \\ &= \sum_{\gamma \in \text{Gal}(K/F)} \dim \text{Hom}_{\mathcal{R}_F(\epsilon)}(\sigma_\epsilon^\gamma, \tau). \end{aligned}$$

If  $\sigma_\epsilon^\gamma$  occurs in  $\tau$ , then  $\sigma_\epsilon^\gamma = \sigma_\epsilon^\delta$ , for some  $\delta \in \mathcal{W}_K$ , and conversely. The sum is therefore effectively taken over  $\gamma \in \mathcal{W}_K \mathcal{W}_{E_\epsilon} / \mathcal{W}_K = \text{Gal}(K/K \cap E_\epsilon)$ , so

$$m_\epsilon(\sigma) = {}_F m_\epsilon(\tau) p / [K \cap E_\epsilon : F].$$

By definition,  $l_\epsilon(\sigma) = [E_\epsilon : F]$  and  ${}_F l_\epsilon(\tau) = [K E_\epsilon : K] = [E_\epsilon : F] / [K \cap E_\epsilon : F]$ . That is,

$$l_\epsilon(\sigma) = {}_F l_\epsilon(\tau) [K \cap E_\epsilon : F].$$

Consequently,

$$d_\epsilon(\sigma) = {}_F d_\epsilon(\tau) p^2 / [K \cap E_\epsilon : F],$$

and, likewise,

$$d_\epsilon^+(\sigma) = {}_F d_\epsilon^+(\tau) p^2 / [K \cap E_\epsilon^+ : F].$$

This proves the first assertion of the lemma.

The quotient  $[K \cap E_\epsilon : F] / [K \cap E_\epsilon^+ : F]$  may take only the values 1,  $p^{\pm 1}$ . It remains to show that the case  $[K \cap E_\epsilon : F] / [K \cap E_\epsilon^+ : F] = p^{-1}$  cannot arise. In other words, we have to show that  $K \cap E_\epsilon = F$  implies  $K \cap E_\epsilon^+ = F$ .

Suppose, therefore, that  $K \cap E_\epsilon = F$  or, as amounts to the same,  $G_\epsilon \mathcal{W}_K = \mathcal{W}_F$ . The restriction of  $\tau$  to  $\mathcal{R}_F(\epsilon)$  is a multiple of  $\sum_\delta \sigma_\epsilon^\delta$ , with  $\delta$  ranging over  $G_\epsilon \cap \mathcal{W}_K \setminus \mathcal{W}_K$ , while  $\sigma \mid \mathcal{R}_F(\epsilon)$  is a multiple of  $\sum_\beta \sigma_\epsilon^\beta$ , with  $\beta \in G_\epsilon \setminus \mathcal{W}_F$ . Our hypothesis  $K \cap E_\epsilon = F$  implies that the natural map  $G_\epsilon \cap \mathcal{W}_K \setminus \mathcal{W}_K \rightarrow G_\epsilon \setminus \mathcal{W}_F$  is bijective. We conclude that  $\sigma \mid \mathcal{R}_F(\epsilon) = p \tau \mid \mathcal{R}_F(\epsilon)$ , whence  $\sigma \mid \mathcal{R}_F^+(\epsilon) = p \tau \mid \mathcal{R}_F^+(\epsilon)$ . Put another way,

$$\frac{d_\epsilon^+(\sigma)}{d_\epsilon(\sigma)} = \frac{{}_F d_\epsilon^+(\tau)}{{}_F d_\epsilon(\tau)},$$

so  $K \cap E_\epsilon^+ = F$ , as required.  $\square$

For  $c$  as in (3.4.2), observe that

$$(3.4.3) \quad \text{Hom}_{\mathcal{R}_F(c) \cap \mathcal{W}_K}(\tau, \tau^\gamma) = 0.$$

Otherwise, the representation  $\tilde{\tau} \otimes \tau^\gamma$  would have an irreducible component  $\lambda$  for which  $\text{Ker } \lambda$  contained  $\mathcal{R}_F(c) \cap \mathcal{W}_K = \mathcal{R}_K(c')$ , where  $c' = \psi_{K/F}(c)$ . In that case,  $\text{Ker } \lambda$  would contain  $\mathcal{R}_K(c'')$ , for some  $c'' < c'$  ([13] 2.1 Proposition 1). That is,  $\text{Hom}_{\mathcal{R}_K(c'')}(\tau, \tau^\gamma) \neq 0$ , contrary to the definition of  $c$ .

**Lemma 3.** *If  $\phi > c$ , then  $w_\phi(\sigma)/_F w_\phi(\tau) = 1$ . If  $c > a$ , then  $w_c(\sigma)/_F w_c(\tau) = p$ .*

*Proof.* Let  $\phi > c$ , so that  $\text{Hom}_{\mathcal{R}_F(\phi)}(\tau, \tau^\gamma) \neq 0$ . It follows that  $\tau$  is  $\mathcal{R}_F(\phi)$ -isomorphic to  $\tau^\gamma$ , for all choices of  $\gamma$ . Therefore  $\sigma|_{\mathcal{R}_F(\phi)}$  is a sum of  $p$  copies of  $\tau|_{\mathcal{R}_F(\phi)}$  and so  $\sigma|_{\mathcal{R}_F^+(\phi)}$  is a sum of  $p$  copies of  $\tau|_{\mathcal{R}_F^+(\phi)}$ . This implies  $w_\phi(\sigma) = _F w_\phi(\tau)$ .

If  $c > a$ , we have  $\text{Hom}_{\mathcal{R}_F(c) \cap \mathcal{W}_K}(\tau, \tau^\gamma) = 0$  while  $\text{Hom}_{\mathcal{R}_F^+(c)}(\tau, \tau^\gamma) \neq 0$ . The second condition implies that  $G_c^+ \mathcal{W}_K = \mathcal{W}_F$ , or  $K \cap E_c^+ = F$  (notation as in the proof of Lemma 2). The first condition implies  $G_c \mathcal{W}_K \neq \mathcal{W}_F$ , or  $K \subset E_c$ . From Lemma 2, we deduce that  $w_c(\sigma)/_F w_c(\tau) = p$ .  $\square$

Consider now the situation at the point  $a$ .

**Lemma 4.** *Let  $\gamma$  generate  $\text{Gal}(K/F)$ .*

- (1) *If  $\text{Hom}_{\mathcal{R}_F^+(a)}(\tau, \tau^\gamma) = 0$ , then  $c > a$  and  $w_a(\sigma)/_F w_a(\tau) = p$ .*
- (2) *If  $\text{Hom}_{\mathcal{R}_F^+(a)}(\tau, \tau^\gamma) \neq 0$ , then  $c = a$  and  $w_a(\sigma)/_F w_a(\tau) = p^2$ .*

*Proof.* The representation  $\sigma|_{\mathcal{R}_F(a)}$  is irreducible and

$$\begin{aligned} \sigma|_{\mathcal{R}_F(a)} &= \sum_{x \in \mathcal{W}_K \setminus \mathcal{W}_F / \mathcal{R}_F(a)} \text{Ind}_{\mathcal{W}_K \cap \mathcal{R}_F(a)}^{\mathcal{R}_F(a)} \tau^x | (\mathcal{W}_K \cap \mathcal{R}_F(a)) \\ &= \text{Ind}_{\mathcal{W}_K \cap \mathcal{R}_F(a)}^{\mathcal{R}_F(a)} \tau | (\mathcal{W}_K \cap \mathcal{R}_F(a)). \end{aligned}$$

It follows that  $\tau$  is irreducible on  $\mathcal{R}_K(a) = \mathcal{R}_F(a) \cap \mathcal{W}_K$ , and that the representations  $\tau^\gamma|_{\mathcal{R}_K(a)}$ ,  $\gamma \in \mathcal{W}_K \setminus \mathcal{W}_F$ , are distinct.

Next,

$$\begin{aligned} \sigma|_{\mathcal{R}_F^+(a)} &= \sum_{x \in \mathcal{W}_K \setminus \mathcal{W}_F / \mathcal{R}_F^+(a)} \text{Ind}_{\mathcal{W}_K \cap \mathcal{R}_F^+(a)}^{\mathcal{R}_F^+(a)} \tau^x | (\mathcal{W}_K \cap \mathcal{R}_F^+(a)) \\ &= \sum_{\gamma \in \mathcal{W}_K \setminus \mathcal{W}_F} \tau^\gamma | \mathcal{R}_F^+(a). \end{aligned}$$



The restrictions  $\tau^\gamma \mid \mathcal{R}_F^+(a)$  are either disjoint or identical. If they are disjoint, then

$$l_a^+(\sigma) = {}_F l_a^+(\tau)p \quad \text{and} \quad m_a^+(\sigma) = {}_F m_a^+(\tau).$$

In this case,

$$d_a^+(\sigma) = {}_F d_a^+(\tau)p \quad \text{and} \quad \tau^\gamma \mid \mathcal{R}_F^+(a) \not\cong \tau \mid \mathcal{R}_F^+(a), \quad \gamma \neq 1.$$

If the  $\tau^\gamma \mid \mathcal{R}_F^+(a)$  are identical, then

$$l_a^+(\sigma) = {}_F l_a^+(\tau), \quad m_a^+(\sigma) = {}_F m_a^+(\tau)p,$$

yielding

$$d_a^+(\sigma) = {}_F d_a^+(\tau)p^2 \quad \text{and} \quad \tau^\gamma \mid \mathcal{R}_F^+(a) \cong \tau \mid \mathcal{R}_F^+(a).$$

Since  $d_a(\sigma) = {}_F d_a(\tau) = 1$ , the lemma follows.  $\square$

We prove the theorem. Part (1) has been done. Part (2) is given by Lemma 3, Lemma 4(1) and (3.4.1). Part (3) follows from (3.4.1) and Lemma 4(2).  $\square$

**3.5.** We continue in the situation of 3.4, except that we now specialize to representations of Carayol type.

**Theorem.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  be of Carayol type and dimension  $p^r$ . Let  $a_\sigma$  be the least jump of the function  $\Sigma_\sigma$ . The largest jump  $z_\sigma$  of  $\Sigma_\sigma$  is then*

$$z_\sigma = c_{K/F}(\sigma) = \frac{\text{sw}(\sigma) - a_\sigma}{p^r}.$$

*Proof.* We proceed by induction on  $r$ . Take  $r = 1$ . We then have  $\Sigma_\sigma(0) = (p-1)\text{sw}(\sigma)/p^2$  (2.4.1) and  $\Sigma_\sigma(x) = x$  for  $x \geq \varsigma_\sigma = \text{sw}(\sigma)/p$ . In particular,  $0 < a_\sigma \leq z_\sigma < \varsigma_\sigma$ . In the region  $0 < x < \varsigma_\sigma$ , the derivative  $\Sigma'_\sigma(x)$  takes the values  $p^{-2}$ , 1 and, possibly,  $p^{-1}$  (as follows from (2.2.2)). If only the values  $p^{-2}$ , 1 occur, then  $a_\sigma$  is the only jump. It lies at the intersection of the lines  $y = p^{-2}x + (p-1)\text{sw}(\sigma)/p^2$  and  $y = x$ , that is  $a_\sigma = \text{sw}(\sigma)/(1+p) = (\text{sw}(\sigma) - a_\sigma)/p$ , as required. If, on the other hand,  $\Sigma'_\sigma$  takes the value  $p^{-1}$  on some interval, then  $z_\sigma$  is given by the intersection of the lines  $y = x$  and  $y - \Sigma_\sigma(a_\sigma) = (x - a_\sigma)/p$ . Since  $\Sigma_\sigma(a_\sigma) = p^{-2}a_\sigma + \Sigma_\sigma(0)$ , the result follows from a quick calculation.

Assume  $r \geq 2$ . From 3.3 Lemma 1 we recall:

**Lemma 1.** *The representation  $\tau$  is absolutely wild of Carayol type and dimension  $p^{r-1}$ .*

We may therefore assume inductively that

$$z_\tau = (\text{sw}(\tau) - a_\tau)/p^{r-1},$$

where  $a_\tau \leq z_\tau$  are the first and last jumps of  $\Sigma_\tau$ . We calculate a list of Swan exponents.

**Lemma 2.**

- (1)  $\text{sw}(\check{\sigma} \otimes \sigma) = (p^r - 1) \text{sw}(\sigma)$ .
- (2)  $\text{sw}(\check{\tau} \otimes \tau) = (p^{r-1} - 1) \text{sw}(\tau)$ .
- (3) *If  $\gamma$  generates  $\text{Gal}(K/F)$ , then  $\text{sw}(\check{\tau} \otimes \tau^\gamma) = p^{r-1}(\text{sw}(\tau) - a_\sigma)$ .*

*Proof.* The representations  $\sigma, \tau$  are of Carayol type, so (1) and (2) are given by (2.4.1) and (2.1.1). As in [13] (2.5.3), set

$$\Delta_K(\rho_1, \rho_2) = \inf \{x > 0 : \text{Hom}_{\mathcal{R}_K(x)}(\rho_1, \rho_2) \neq 0\}.$$

Thus ([13] (3.1.4))

$$\frac{\text{sw}(\check{\rho}_1 \otimes \rho_2)}{\dim(\rho_1) \dim(\rho_2)} = \Sigma_{\rho_1}(\Delta_K(\rho_1, \rho_2)), \quad \rho_i \in \widehat{\mathcal{W}}_K.$$

We started the proof of 3.4 Theorem by observing that, in effect,  $\Delta_K(\tau, \tau^\gamma)$  is independent of  $\gamma \in \text{Gal}(K/F)$ ,  $\gamma \neq 1$ . It follows that  $\text{sw}(\check{\tau} \otimes \tau^\gamma)$  does not depend on  $\gamma$ . With this in mind, we apply the induction formula for the Swan conductor to the relations

$$\begin{aligned} \check{\tau} \otimes \sigma \mid \mathcal{W}_K &= \sum_{\gamma \in \text{Gal}(K/F)} \check{\tau} \otimes \tau^\gamma, \\ \check{\sigma} \otimes \sigma &= \text{Ind}_{K/F}(\check{\tau} \otimes \sigma \mid \mathcal{W}_K). \end{aligned}$$

Since  $w_{K/F} = (p-1)a_\sigma$  we get, for any  $\gamma \neq 1$ ,

$$(p-1) \text{sw}(\check{\tau} \otimes \tau^\gamma) = \text{sw}(\check{\sigma} \otimes \sigma) - \text{sw}(\check{\tau} \otimes \tau) - p^{2r-1}(p-1)a_\sigma,$$

whence (3) follows.  $\square$

*Remark.* The formulæ in parts (1) and (2) of Lemma 2 rely ultimately on the conductor formula of [14]. This is the only intervention from the GL-side in the proofs of the theorems of 3.4 and 3.5. It is, however, crucial.

The definition of  $c = c_{K/F}$  in (3.4.2) gives  $\psi_{K/F}(c) = \Delta_K(\tau, \tau^\gamma)$ . Since  $c \geq a_\sigma$  (3.4 Theorem (1)), we have  $\psi_{K/F}(c) = a_\sigma + p(c - a_\sigma)$ .

**Lemma 3.** *If  $\gamma \in \text{Gal}(K/F)$ ,  $\gamma \neq 1$ , then  $\Delta_K(\tau, \tau^\gamma) \geq z_\tau$ . Equality holds here if and only if  $a_\sigma = a_\tau$ .*

*Proof.* The relation  $\Sigma_\tau(\Delta_K(\tau, \tau^\gamma)) = p^{2-2r} \text{sw}(\check{\tau} \otimes \tau^\gamma)$  reduces us to proving

$$\text{sw}(\check{\tau} \otimes \tau^\gamma) \geq p^{2r-2} \Sigma_\tau(z_\tau).$$

Since  $z_\tau$  is the last jump of  $\Sigma_\tau$ , we have  $\Sigma_\tau(y) = y$ , for  $y > z_\tau$ . In particular,  $\Sigma_\tau(z_\tau) = z_\tau$ . The inductive hypothesis therefore yields

$$p^{2r-2} \Sigma_\tau(z_\tau) = p^{r-1}(\text{sw}(\tau) - a_\tau).$$

On the other hand,  $\text{sw}(\check{\tau} \otimes \tau^\gamma) = p^{r-1} \text{sw}(\tau) - p^{r-1} a_\sigma$  by Lemma 2(3). By 3.4 Lemma 1, we have  $a_\sigma \leq a_\tau$  whence the result follows.  $\square$

**Lemma 4.** *The element  $c = c_{K/F}(\sigma)$  satisfies  $c = z_\sigma \geq \varphi_{K/F}(z_\tau)$ .*

*Proof.* By definition and 3.3 Lemma 2, the number  $\varphi_{K/F}(z_\tau)$  is the infimum of  $\epsilon > 0$  such that  $\tau \mid \mathcal{R}_F(\epsilon)$  is a multiple of a character. Lemma 3 gives

$$(3.5.1) \quad c = \varphi_{K/F}(\Delta_K(\tau, \tau^\gamma)) \geq \varphi_{K/F}(z_\tau)$$

while, on the other hand,  $c$  is the infimum of numbers  $\epsilon$  such that  $\tau \mid \mathcal{R}_F(\epsilon) \cap \mathcal{W}_K \cong \tau^\gamma \mid \mathcal{R}_F(\epsilon) \cap \mathcal{W}_K$ . Thus (3.5.1) implies that  $c$  is the infimum of numbers  $\epsilon$  such that  $\sigma \mid \mathcal{R}_F(\epsilon)$  is a multiple of a character. That is,  $c = z_\sigma \geq \varphi_{K/F}(z_\tau)$ , as required.  $\square$

Lemma 4 yields the first assertion of the theorem. We prove the second. To complete the induction, we have to show that

$$c = z_\sigma = p^{-r}(\text{sw}(\sigma) - a_\sigma).$$

Abbreviating  $\Delta = \Delta_K(\tau, \tau^\gamma)$ , (3.5.1) asserts that

$$(3.5.2) \quad \psi_{K/F}(c) = a_\sigma + p(c - a_\sigma) = \Delta.$$

We have  $\Sigma_\tau(y) = y$ , for  $y \geq z_\tau$ , while Lemma 3 gives  $\Delta \geq z_\tau$ . So,

$$\Delta = \Sigma_\tau(\Delta) = \text{sw}(\check{\tau} \otimes \tau^\gamma) / p^{2r-2} = p^{1-r}(\text{sw}(\tau) - a_\sigma).$$

Combining with (3.5.2), we get

$$p^r c = \text{sw}(\tau) + (p^r - p^{r-1}) a_\sigma.$$

However,  $\text{sw}(\tau) = \text{sw}(\sigma) - p^{r-1}(p-1)a_\sigma$ , whence

$$(3.5.3) \quad z_\sigma = c = p^{-r}(\text{sw}(\sigma) - a_\sigma),$$

as required.  $\square$

Keeping the notation of the theorem, we exhibit a consequence.

**Corollary 1.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  be of Carayol type and degree  $p^r$ ,  $r \geq 1$ . Set  $a = a_\sigma$ . If  $w_a(\sigma)/_F w_a(\tau) = p^2$ , then  $a$  is the unique jump of the function  $\Sigma_\sigma$ .*

*Proof.* Lemma 4(2) of 3.4 implies  $c = a_\sigma$ . We have just shown that  $c = z_\sigma$ . The function  $\Sigma_\sigma$  thus has a unique jump.  $\square$

*Remark.* The conclusion of the corollary has strong implications for the structure of the representation  $\sigma$ : see 8.3 Proposition 2 below.

To finish, we note that, because of (2.2.3), the theorem and its corollary apply equally to totally wild representations that are not absolutely wild. In particular,

**Corollary 2.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type and dimension  $p^r$ . If  $a_\sigma$  and  $z_\sigma$  are the first and last jumps of the function  $\Sigma_\sigma$  respectively, they are related by*

$$z_\sigma = \frac{\text{sw}(\sigma) - a_\sigma}{p^r}.$$

**3.6.** We start the proof of the functional equation (3.1.1). The argument occupies the rest of the section.

In 3.4, 3.5, we effectively worked with decomposition functions. We must now pass to Herbrand functions. To avoid the need for more notation, we work with endo-classes. Nonetheless, the underlying technique is entirely Galois-theoretic and could be phrased in those terms. We start with the necessary translation.

**Proposition.** *Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$ ,  $r \geq 1$ . If  $a_\Theta \leq z_\Theta$  are the first and last jumps of  $\Psi_\Theta$ , then*

$$(3.6.1) \quad z_\Theta = \varsigma_\Theta - a_\Theta/p^r = \varsigma_\Theta - \Psi_\Theta(a_\Theta).$$

*Proof.* There exists an irreducible cuspidal representation  $\pi$  of  $\text{GL}_{p^r}(F)$  that contains a simple character of endo-class  $\Theta$ . The representation  $\sigma = {}^L\pi$  is therefore totally wild of dimension  $p^r$ . Moreover,  $\text{sw}(\sigma) = p^r \varsigma_\sigma = p^r \varsigma_\Theta$  is not divisible by  $p$ , so  $\sigma$  is of Carayol type. The formula in part (1) of 2.4 Proposition implies that the functions  $\Psi_\Theta$ ,  $\Sigma_\sigma$  have the same jumps. In particular,  $a_\Theta = a_\sigma$  and  $z_\Theta = z_\sigma$ . The first equality in (3.6.1) thus follows from 3.5 Theorem. In the range  $0 < x < a_\Theta$ , we have  $\Psi'_\Theta(x) = p^{-r}$  and so  $a_\Theta/p^r = \Psi_\Theta(a_\Theta)$ , as required for the second equality.  $\square$

**3.7.** Let  $\Theta \in \mathcal{E}(F)$  be totally wild. Say that  $\Theta$  is *absolutely wild* if there exists  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  such that  ${}^L\Theta = [\sigma]_0^+$ . The relation  $[\sigma]_0^+ = {}^L\Theta$  determines  $\sigma$  up to tensoring with a tame character of  $\mathcal{W}_F$  [12] 1.3 Proposition. So, if one choice of  $\sigma$  is absolutely wild, then all are.

For given  $\Theta$ , there surely exists a finite tame extension  $T/F$  so that the unique  $T/F$ -lift  $\Theta^T$  of  $\Theta$  is absolutely wild. We have  $\varsigma_{\Theta^T} = e(T|F)\varsigma_\Theta$ . From (2.2.3) we deduce that if (3.1.1) holds for  $\Theta^T$  it also holds for  $\Theta$ . We therefore proceed on the basis that the given endo-class  $\Theta$  is *absolutely wild*.

For the next result, take  $\Theta \in \mathcal{E}^C(F)$ , absolutely wild of degree  $p^r$ . Choose  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  so that  $\sigma|_{\widehat{\mathcal{P}}_F} = {}^L\Theta$ . Define  $a = a_\sigma$ ,  $K/F$  and  $\tau$ , relative to  $\sigma$ , as in 3.3 Lemma 1. Let  $c = c_{K/F}(\sigma)$  as in (3.4.2), and note that  $a = a_\Theta$ .

**Proposition.** *There exists a unique  $\Upsilon \in \mathcal{E}(K)$  such that  $[\tau]_0^+ = {}^L\Upsilon$ . If  $r \geq 2$ , the endo-class  $\Upsilon$  is absolutely wild of degree  $p^{r-1}$  while, otherwise,  $\deg \Upsilon = 1$ . In either case, it satisfies*

$$\Psi_\Theta(x) = p^{-1}\Psi_\Upsilon(\psi_{K/F}(x)), \quad 0 \leq x \leq c.$$

*Proof.* The existence and uniqueness of  $\Upsilon$  are clear. If  $r \geq 2$ , then  $\tau$  is absolutely wild, whence so is  $\Upsilon$ . In the region  $0 \leq x \leq a$ , we have  $\Psi_\Theta(x) = p^{-r}x$  while  $\Psi_\Upsilon(\psi_{K/F}(x)) = \Psi_\Upsilon(x) = p^{1-r}x$ . The required relation therefore holds in this range. In the case  $a = c$ , there is nothing left to do so we assume  $a < c$ .

If  $a < x < c$ , 3.4 Theorem gives  $w_x(\sigma) = {}_Fw_x(\tau)$ . In other words, the ratio of the derivatives of  $\Psi_\Theta$  and  $\Psi_\Upsilon \circ \psi_{K/F}$  is constant, and equal to  $p$  in this range. The result follows.  $\square$

**3.8.** We prove (3.1.1). Let  $\Theta \in \mathcal{E}^C(F)$  be absolutely wild, of degree  $p^r$ . We first dispose of a singular case.

**Proposition.** *Suppose that  $\Psi_\Theta$  has a unique jump  $a$ . The functional equation (3.1.1) then holds for  $\Theta$  and  $a = p^r\varsigma_\Theta/(1+p^r)$ .*

*Proof.* Appealing to 2.4 Proposition part (3), the graph of  $\Psi_\Theta$ , in the range  $0 \leq x \leq \varsigma_\Theta$ , comprises only segments of the two lines  $y = p^{-r}x$ ,  $y = p^r x - (p^r - 1)\varsigma_\Theta$ . The latter has slope  $p^r$  and passes through the point  $(\varsigma_\Theta, \varsigma_\Theta)$ . These two lines intersect at the point  $(a, p^{-r}a)$ , where  $a = p^r\varsigma_\Theta/(1+p^r)$ . Using the Symmetry formulation of 3.1, the result is clear in this case.  $\square$

We assume henceforth that  $\Psi_\Theta$  has at least two jumps and proceed by induction on  $r$ . Suppose  $r = 1$ . In this case,  $\Psi_\Theta$  has exactly two jumps, and they are related as in 3.6 Proposition. The graph consists of segments of the two lines  $y = p^{-1}x$ ,  $y = px - (p-1)\varsigma_\Theta$  and a non-empty segment of a third line of slope 1. Using the symmetry formulation, the result is clear in this case.

Suppose therefore that  $r \geq 2$  and that  $\Psi_\Theta$  has at least two distinct jumps. Let  $a = a_\Theta$  be the least jump. There exists a character  $\chi$  of  $F^\times$ , of Swan exponent  $a$  and order  $p$ , such that  $\chi\Theta = \Theta$  (as follows from 3.3 Lemma 1). View  $\chi$  as a character of  $\mathcal{W}_F$  and let  $\mathcal{W}_K = \text{Ker } \chi$ . Take  $\Upsilon \in \mathfrak{E}^C(K)$  as in 3.7 Proposition. By inductive hypothesis,

$$\varsigma_\Upsilon - y = \Psi_\Upsilon(\varsigma_\Upsilon - \Psi_\Upsilon(y)), \quad 0 \leq y \leq \varsigma_\Upsilon.$$

Let  $z = z_\Theta$  be the largest jump of  $\Psi_\Theta$  and  $z_K$  that of  $\Psi_\Upsilon \circ \psi_{K/F}$ . It follows from (3.5.1) that  $z_K \leq z$ . In the range  $z < x < \varsigma_\Theta$ , we have

$$\Psi_\Theta(x) = \varsigma_\Theta - p^r(\varsigma_\Theta - x).$$

Also,  $\varsigma_\Theta - x < \varsigma_\Theta - z = a/p^r$ , by 3.5 Theorem. Therefore

$$\Psi_\Theta(\varsigma_\Theta - \Psi_\Theta(x)) = \Psi_\Theta(p^r(\varsigma_\Theta - x)) = \varsigma_\Theta - x,$$

as desired. If, on the other hand,  $0 < x < a$ , then  $\Psi_\Theta(x) = x/p^r$ , whence

$$\varsigma_\Theta - \Psi_\Theta(x) = \varsigma_\Theta - x/p^r > \varsigma_\Theta - a/p^r = z.$$

Therefore  $\Psi_\Theta(\varsigma_\Theta - \Psi_\Theta(x)) = \varsigma_\Theta - x$ .

It remains to treat the range  $a < x < z$ . Here,  $\varsigma_\Theta - \Psi_\Theta(x) < \varsigma_\Theta - \Psi_\Theta(a) = \varsigma_\Theta - a/p^r = z$ . We may therefore apply 3.7 Proposition and (3.5.3) to get

$$\Psi_\Theta(\varsigma_\Theta - \Psi_\Theta(x)) = p^{-1} \Psi_\Upsilon(\psi_{K/F}(\varsigma_\Theta - \Psi_\Theta(x))).$$

We have

$$\Psi_\Theta(x) < \Psi_\Theta(z) = \varsigma_\Theta - p^r(\varsigma_\Theta - z) = \varsigma_\Theta - a.$$

That is,  $\varsigma_\Theta - \Psi_\Theta(x) > a$ . It follows that

$$\begin{aligned} \psi_{K/F}(\varsigma_\Theta - \Psi_\Theta(x)) &= \psi_{K/F}(\varsigma_\Theta) - p\Psi_\Theta(x) \\ &= \varsigma_\Upsilon - p\Psi_\Theta(x). \end{aligned}$$

Therefore

$$\begin{aligned}\Psi_{\Theta}(\varsigma_{\Theta} - \Psi_{\Theta}(x)) &= p^{-1} \Psi_{\mathcal{R}}(\varsigma_{\mathcal{R}} - p\Psi_{\Theta}(x)) \\ &= p^{-1} \Psi_{\mathcal{R}}(\varsigma_{\mathcal{R}} - \Psi_{\mathcal{R}}(\psi_{K/F}(x))) \\ &= p^{-1}(\varsigma_{\mathcal{R}} - \psi_{K/F}(x)),\end{aligned}$$

applying the inductive hypothesis at the last step. Finally,

$$p^{-1}(\varsigma_{\mathcal{R}} - \psi_{K/F}(x)) = p^{-1}(\psi_{K/F}(\varsigma_{\Theta}) - \psi_{K/F}(x)) = \varsigma_{\Theta} - x,$$

and the proof is complete.  $\square$

#### 4. Symmetry and the bi-Herbrand function

We turn attention to the GL-side. Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$  (notation of 2.3). In particular,  $r \geq 1$ . We observed in 3.1 that the functional equation (3.1.1) can be interpreted as a symmetry property of the graph of  $\Psi_{\Theta}$ . This leads us to define a family of more transparent “bi-Herbrand functions” with the same properties of symmetry and convexity. Our objective, realized in section 7, is to calculate  $\Psi_{\Theta}$  as an explicit bi-Herbrand function. However, 4.6 Example at the end of the section does exactly that in a substantial family of cases.

**4.1.** We draw out some useful features of the graph  $y = \Psi_{\Theta}(x)$ . For  $\lambda > 0$ , let  $i_{\lambda}$  be the reflection in the line  $x+y = \lambda$ . That is,

$$i_{\lambda} : (x, y) \mapsto (\lambda - y, \lambda - x).$$

**Proposition.** *Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$  and abbreviate  $\varsigma = \varsigma_{\Theta}$ .*

- (1) *The graph  $y = \Psi_{\Theta}(x)$ ,  $0 \leq x \leq \varsigma$ , is stable under the reflection  $i_{\varsigma}$ .*
- (2) *There is a unique point  $c_{\Theta}$  such that  $c_{\Theta} + \Psi_{\Theta}(c_{\Theta}) = \varsigma$ . The following conditions are equivalent.*
  - (a) *The point  $c_{\Theta}$  is not a jump of  $\Psi_{\Theta}$ .*
  - (b) *The function  $\Psi_{\Theta}$  has an even number of jumps.*
  - (c) *The function  $\Psi'_{\Theta}$  takes the value 1 on a non-empty open subset of the region  $0 < x < \varsigma$ .*
  - (d) *The set  $I$  of  $x$  for which  $\Psi'_{\Theta}(x) = 1$  is an open interval containing  $c_{\Theta}$ .*

(3) *If the conditions (2)(a)–(d) hold, then*

$$\Psi_{\Theta}(x) = x - 2c_{\Theta} + \varsigma, \quad x \in I.$$

(4) *Let  $0 \leq x \leq \varsigma$ . In all cases,  $\Psi'_{\Theta}(x) \leq 1$  if  $x + \Psi_{\Theta}(x) \leq \varsigma$ , while  $\Psi'_{\Theta}(x) \geq 1$  if  $x + \Psi_{\Theta}(x) \geq \varsigma$ .*

*Proof.* Part (1) has been proved in 3.1, as a consequence of (3.1.1). The function  $\Psi_{\Theta}$  is strictly increasing, giving the first assertion in (2). The equivalence of (a), (b) and (d) follows from the symmetry of part (1). Suppose (c) holds, and let  $I$  be the set of  $x$ ,  $0 < x < \varsigma$ , for which  $\Psi'_{\Theta}(x) = 1$ . The convexity of  $\Psi_{\Theta}$  implies that  $I$  is an interval and symmetry implies  $c_{\Theta} \in I$ . Thus (c) implies (d), and surely (d) implies (c).

In part (3), there is a neighbourhood of  $c_{\Theta}$  on which  $\Psi_{\Theta}(x) = x - b$ , for some constant  $b$ . Thus  $\varsigma = c_{\Theta} + \Psi_{\Theta}(c_{\Theta}) = 2c_{\Theta} - b$ , whence  $b = 2c_{\Theta} - \varsigma$ , as required. Part (4) follows from the convexity of  $\Psi_{\Theta}$  and the symmetry property of (1).  $\square$

*Remark.* The function  $\Psi_{\Theta}$  is continuous and strictly increasing. The condition  $x + \Psi_{\Theta}(x) \leq \varsigma$  of part (4) is therefore equivalent to  $x \leq c_{\Theta}$ .

We frequently use the following simple observation, so we exhibit it as a corollary.

**Corollary.** *The function  $\Psi_{\Theta}$  has an odd number of jumps if and only if  $c_{\Theta}$  is a jump. In that case,  $c_{\Theta}$  is the middle one.*

*Proof.* The reflection  $i_{\varsigma}$  stabilizes the set of jumps of  $\Psi_{\Theta}$  but fixes the point  $(c_{\Theta}, \Psi_{\Theta}(c_{\Theta}))$ .  $\square$

**4.2.** We construct a family of  $i_{\varsigma}$ -symmetric functions using more transparent data. They have properties analogous to those in 4.1 Proposition. To specify them, we need two families of auxiliary functions defined using the classical Herbrand functions  $\psi_{E/F}$ ,  $\varphi_{E/F}$  of section 1.

**Definition.** Let  $E/F$  be a totally ramified field extension of degree  $p^r$ ,  $r \geq 1$ . Let  $\varsigma = m/p^r$ , where  $m$  is a positive integer not divisible by  $p$ . Define

$$(4.2.1) \quad \begin{aligned} \Psi_{(E/F, \varsigma)}^{\times}(x) &= p^{-r} \psi_{E/F}(x), \\ \Psi_{(E/F, \varsigma)}^{+}(x) &= \varsigma - \varphi_{E/F}(p^r(\varsigma - x)), \end{aligned} \quad 0 \leq x \leq \varsigma.$$

The functions  $\Psi_{(E/F, \varsigma)}^{\times}$ ,  $\Psi_{(E/F, \varsigma)}^{+}$  are continuous, strictly increasing, convex and piecewise linear in the region  $0 \leq x \leq \varsigma$ . They have only finitely many jumps.



**Lemma.**

- (1) The functions  $\Psi_{(E/F, \varsigma)}^\times, \Psi_{(E/F, \varsigma)}^+$  satisfy

$$\begin{aligned} \varsigma - x &= \Psi_{(E/F, \varsigma)}^+(\varsigma - \Psi_{(E/F, \varsigma)}^\times(x)) \\ &= \Psi_{(E/F, \varsigma)}^\times(\varsigma - \Psi_{(E/F, \varsigma)}^+(x)). \end{aligned}$$

- (2) There is a unique point  $c = c_{(E/F, \varsigma)}$  such that  $c + \Psi_{(E/F, \varsigma)}^\times(c) = \varsigma$ . It further satisfies  $c + \Psi_{(E/F, \varsigma)}^+(c) = \varsigma$ .  
 (3) Let  $j_\infty = j_\infty(E|F)$  and suppose  $j_\infty < \varsigma$ . The point  $j_\infty$  is the largest jump of  $\Psi_{(E/F, \varsigma)}^\times$  and

$$(4.2.2) \quad \bar{j}_\infty = \varsigma - \Psi_{(E/F, \varsigma)}^\times(j_\infty)$$

is the least jump of  $\Psi_{(E/F, \varsigma)}^+$ . If  $j_\infty < c$ , then  $c < \bar{j}_\infty < \varsigma$ .

*Proof.* Part (1) follows from a simple manipulation of the definition (4.2.1). In (2), the function  $\Psi_{(E/F, \varsigma)}^\times$  is strictly increasing and  $\Psi_{(E/F, \varsigma)}^\times(0) = 0$ , giving the first assertion. For the second, we abbreviate the notation in the obvious way. From the definitions,  $\varsigma - c = \Psi^+(\varsigma - \Psi^\times(c)) = \Psi^+(c)$ , as required. The graphs  $y = \Psi_{(E/F, \varsigma)}^\times(x)$ ,  $y = \Psi_{(E/F, \varsigma)}^+(x)$  are interchanged by the involution  $i_\varsigma$ , whence (3) follows.  $\square$

We define the *bi-Herbrand function*  ${}^2\Psi_{(E/F, \varsigma)}$  by

$$(4.2.3) \quad {}^2\Psi_{(E/F, \varsigma)}(x) = \max\{\Psi_{(E/F, \varsigma)}^\times(x), \Psi_{(E/F, \varsigma)}^+(x)\}, \quad 0 \leq x \leq \varsigma.$$

When speaking of the jumps of  ${}^2\Psi_{(E/F, \varsigma)}$ , we mean the discontinuities of its derivative in the region  $0 < x < \varsigma$ .

**Proposition.** Let  $j_\infty = j_\infty(E|F)$  and write  $c = c_{(E/F, \varsigma)}$ , as in the lemma.

- (1) The function  ${}^2\Psi_{(E/F, \varsigma)}$  is continuous, strictly increasing, piecewise linear and convex, with only finitely many jumps. The graph  $y = {}^2\Psi_{(E/F, \varsigma)}(x)$  is symmetric with respect to the line  $x + y = \varsigma$ .  
 (2) Suppose  $j_\infty \geq c$ . The function  ${}^2\Psi_{(E/F, \varsigma)}$  has an odd number of jumps, of which  $c$  is the middle one. The derivative  ${}^2\Psi'_{(E/F, \varsigma)}$  does not take the value 1. Moreover,

$${}^2\Psi_{(E/F, \varsigma)}(x) = \begin{cases} \Psi_{(E/F, \varsigma)}^\times(x) > \Psi_{(E/F, \varsigma)}^+(x), & 0 < x < c, \\ \Psi_{(E/F, \varsigma)}^+(x) > \Psi_{(E/F, \varsigma)}^\times(x), & c < x < \varsigma. \end{cases}$$

(3) Suppose  $j_\infty < c$ . Defining  $\bar{j}_\infty$  as in (4.2.2), we have  $j_\infty < c < \bar{j}_\infty$ .

(a) If  $j_\infty < x < \bar{j}_\infty$ , then

$$\begin{aligned} {}^2\Psi'_{(E/F, \varsigma)}(x) &= \Psi^{\times'}_{(E/F, \varsigma)}(x) = \Psi^{+'}_{(E/F, \varsigma)}(x) = 1, \\ {}^2\Psi_{(E/F, \varsigma)}(x) &= \Psi^\times_{(E/F, \varsigma)}(x) = \Psi^+_{(E/F, \varsigma)}(x) = x - p^{-r}w_{E/F}. \end{aligned}$$

(b) If  $0 < x < j_\infty$ , then  $\Psi^{+'}_{(E/F, \varsigma)}(x) = 1 > \Psi^{\times'}_{(E/F, \varsigma)}(x)$  and

$${}^2\Psi_{(E/F, \varsigma)}(x) = \Psi^\times_{(E/F, \varsigma)}(x) > \Psi^+_{(E/F, \varsigma)}(x).$$

(c) If  $\bar{j}_\infty < x < \varsigma$ , then  $\Psi^{\times'}_{(E/F, \varsigma)}(x) = 1 < \Psi^{+'}_{(E/F, \varsigma)}(x)$  and

$${}^2\Psi_{(E/F, \varsigma)}(x) = \Psi^+_{(E/F, \varsigma)}(x) > \Psi^\times_{(E/F, \varsigma)}(x).$$

In particular,  ${}^2\Psi_{(E/F, \varsigma)}$  has an even number of jumps.

*Proof.* In (1), only convexity requires comment, and that is obvious from parts (2) and (3).

The index  $(E/F, \varsigma)$  will be constant throughout, so we omit it for the rest of this argument. We have  $\Psi^\times(c) = \Psi^+(c) = {}^2\Psi(c)$ . We examine the functions in a small neighbourhood of  $x = c$ . The values of  $\Psi^{\times'}(x)$  are of the form  $p^{-s}$ , and those of  $\Psi^{+'}(x)$  are  $p^s$ , for various integers  $s$  such that  $0 \leq s \leq r$ . In part (2), the left derivative of  $\Psi^\times$  at  $c$  is, at most,  $p^{-1}$ , while the right derivative of  $\Psi^+$  at  $c$  is at least  $p$ . So,  $c$  is a jump of  ${}^2\Psi$ . The other assertions in (2) follow from the convexity of the functions  $\Psi^\times$  and  $\Psi^+$ .

In part (3), the functions  $\Psi^\times$ ,  $\Psi^+$  agree, and have derivative 1, on the interval  $j_\infty < x < \bar{j}_\infty$  (which contains  $c$ ). The derivative relations are clear from the definitions, and imply the main points readily.  $\square$

*Remark.* By 1.6 Proposition, the condition  $j_\infty \geq c$  amounts to

$$j_\infty + \Psi^\times_{(E/F, \varsigma)}(j_\infty) = 2j_\infty - p^{-r}w_{E/F} \geq \varsigma.$$

By 1.6 Corollary, this will hold if  $w_{E/F} \geq m(p^r - 1)/(p^r + 1)$ .

**4.3.** We restate 2.6 Proposition in terms of the bi-Herbrand function.

**Proposition.** *Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  be a realization of  $\Theta$  on a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M_{p^r}(F)$ . If  $\varsigma = \varsigma_\Theta = m/p^r$  and  $E = F[\alpha]$  then*

$$\begin{aligned}\Psi_\Theta(x) &= {}^2\Psi_{(E/F, \varsigma)}(x) = \Psi_{(E/F, \varsigma)}^\times(x), \quad 0 \leq x \leq \varsigma/2, \\ \Psi_\Theta(x) &= {}^2\Psi_{(E/F, \varsigma)}(x) = \Psi_{(E/F, \varsigma)}^+(x), \quad \varsigma/2 \leq \Psi_{(E/F, \varsigma)}^+(x) \leq \varsigma.\end{aligned}$$

*Proof.* The first assertion combines 2.6 Proposition with 4.2 Proposition. The second follows by symmetry for  $\Psi_\Theta$  and  ${}^2\Psi_{(E/F, \varsigma)}$ .  $\square$

**4.4.** We record the effect of tame lifting on these functions.

**Proposition.** *Let  $E/F$  be totally ramified of degree  $p^r$  and let  $\varsigma = m/p^r$ , for a positive integer  $m$  not divisible by  $p$ . If  $K/F$  is a finite tame extension and  $e = e(K|F)$ , then*

$$\begin{aligned}\Psi_{(E/F, \varsigma)}^\times(x) &= \Psi_{(EK/K, e\varsigma)}^\times(ex)/e, \\ \Psi_{(E/F, \varsigma)}^+(x) &= \Psi_{(EK/K, e\varsigma)}^+(ex)/e, \quad 0 \leq x \leq \varsigma. \\ {}^2\Psi_{(E/F, \varsigma)}(x) &= {}^2\Psi_{(EK/K, e\varsigma)}(ex)/e,\end{aligned}$$

*Proof.* This combines the definitions (4.2.1), (4.2.3) with 1.2 Lemma.  $\square$

**4.5.** The second assertion of 4.3 Proposition determines  $\Psi_\Theta$  where  $\Psi_\Theta(x) > \varsigma/2$ . That has already been done in 2.7 Proposition, but in a rather different way. Reconciliation of the two approaches reveals a fundamental property of  $\Psi_{(E/F, \varsigma)}^+$ . See 2.5 Definition for the notion of “twisting datum” .

**Proposition.** *Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M_{p^r}(F)$ , in which  $E = F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $m$  is not divisible by  $p$ . Set  $\varsigma = m/p^r$ . If  $(k, c, \chi)$  is a twisting datum over  $F$  such that  $k < m/p^r$  is not a jump of  $\Psi_{(E/F, \varsigma)}^+$  then*

$$\Psi_{(E/F, \varsigma)}^+(k) = t/p^r,$$

where  $t$  is the least integer for which the congruence

$$u^{-1}\alpha u \equiv \alpha + c \pmod{\mathfrak{p}^{-t}}$$

admits a solution  $u \in U_{\mathfrak{a}}^1$ .

*Proof.* Assume initially that  $2t > m$ . For comparison purposes, choose  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  and let  $\Theta$  be the endo-class of  $\theta$ . Thus  $\Theta$  is totally wild and of Carayol type. By 4.3 Proposition,  $k$  is not a jump of  $\Psi_\Theta$  and so, by 2.7 Proposition,  $t/p^r = \Psi_\Theta(k) = \Psi_{(E/F, \varsigma)}^+(k)$ . Because of the jump condition,  $t$  depends on  $k$  but not on the element  $c \in \mathfrak{p}_F^{-k} \setminus \mathfrak{p}_F^{1-k}$ .

We now admit the possibility  $2t \leq m$ . The integer  $t$  depends on  $\alpha$  and  $c$ , so we define a function  $T(\alpha, c) = p^{-r}t$  where, as before,  $t$  is the least integer for which (4.5.1) admits a solution. Let  $n$  be a positive integer and take  $\nu \in F$  with  $v_F(\nu) = -n$ . Thus  $[\mathfrak{a}, m+p^r n, 0, \nu\alpha]$  is a simple stratum in  $M_{p^r}(F)$ . The congruences

$$\begin{aligned} u^{-1}\alpha u &\equiv \alpha + c \pmod{\mathfrak{p}^{-t}}, \\ u^{-1}\nu\alpha u &\equiv \nu(\alpha + c) \pmod{\mathfrak{p}^{-(t+p^r n)}} \end{aligned}$$

have the same sets of solutions  $u \in U_{\mathfrak{a}}^1$ . Consequently,

$$T(\nu\alpha, \nu c) = T(\alpha, c) + n.$$

Provided  $2T(\nu\alpha, \nu c) > \varsigma + n$ , we therefore have

$$T(\nu\alpha, \nu c) = \Psi_{(E/F, \varsigma+n)}^+(k+n).$$

The definition of the functions  $\Psi_{(E/F, \varsigma)}^+$  implies

$$\Psi_{(E/F, \varsigma+n)}^+(x+n) = \Psi_{(E/F, \varsigma)}^+(x) + n,$$

so  $k+n$  is not a jump of  $\Psi_{(E/F, \varsigma+n)}^+(x+n)$ . The condition  $2T(\nu\alpha, \nu c) > \varsigma + n$  thus reduces to  $2T(\alpha, c) > \varsigma - n$ . So, for integers  $k = -v_F(c)$  in that range, we have  $\Psi_{(E/F, \varsigma)}^+(k) = T(\alpha, c)$ . Allowing  $n$  to increase without bound, we see that  $\Psi_{(E/F, \varsigma)}^+(k) = T(\alpha, c)$ , for all integers  $k$  that are not jumps of  $\Psi_{(E/F, \varsigma)}^+$ .  $\square$

*Remark.* The relation between the function  $\Psi_{(E/F, \varsigma)}^+$  and intertwining properties of simple strata was observed in more general work of E.-W. Zink [39], [40] on a corresponding problem in  $F$ -division algebras.

**4.6.** To finish the section with an example, we calculate  $\Psi_\Theta$  in a large family of cases (but see 5.10 Proposition for a stronger result).

**Example.** Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$ ,  $r \geq 1$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  be a realization of  $\Theta$  on a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M_{p^r}(F)$ . Write  $\varsigma = \varsigma_\Theta = m/p^r$  and  $E = F[\alpha]$ . If  $j_\infty(E|F) < \varsigma/2$ , then

$$\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x), \quad 0 \leq x \leq \varsigma.$$

*Proof.* By 4.3 Proposition,  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x)$  for  $0 \leq x \leq \varsigma/2$ . Likewise for  $\varsigma - \Psi_\Theta(\varsigma/2) \leq x \leq \varsigma$  by symmetry. In particular,  $\Psi'_\Theta(x) = {}^2\Psi'_{(E/F, \varsigma)}(x) = 1$  for  $j_\infty < x < \varsigma/2$ . Thus 4.1 Proposition (2) applies. It shows that  $\Psi'_\Theta(x) = 1$  on the set  $j_\infty < x < \varsigma - \Psi_\Theta(j_\infty)$ . The same argument, using 4.2 Proposition, applies to  ${}^2\Psi_{(E/F, \varsigma)}$ , whence  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x)$  on this range. Overall,  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x)$  for  $0 \leq x \leq \varsigma$ .  $\square$

**Gloss.** The hypothesis  $j_\infty < \varsigma/2$  holds if  $w_{E/F} < (p-1)m/2p$ .

*Proof.* By 1.6 Corollary,  $j_\infty \leq p^{1-r}w_{E/F}/(p-1)$ .  $\square$

## 5. Characters of restricted level

Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M = M_{p^r}(F)$ ,  $r \geq 1$ , satisfying the usual conditions:

- (1)  $E = F[\alpha]/F$  is totally ramified of degree  $p^r$  and
- (2)  $m$  is not divisible by  $p$ .

Let  $\|\mathcal{C}(\mathfrak{a}, \alpha)\|$  be the set of endo-classes of simple characters  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ . Thus any  $\Theta \in \|\mathcal{C}(\mathfrak{a}, \alpha)\|$  lies in  $\mathcal{E}^C(F)$  and has degree  $p^r$ . In this section, we fix  $\alpha$  and identify a set of  $\Theta \in \|\mathcal{C}(\mathfrak{a}, \alpha)\|$  for which  $\Psi_\Theta = {}^2\Psi_{(F[\alpha]/F, m/p^r)}$ . This will be the set called  $\mathcal{L}_\alpha$  in the Introduction.

**5.1.** We recall, in the special case to hand, some of the machinery of [15] Chapter 1. Let  $\mathfrak{p}$  be the Jacobson radical of  $\mathfrak{a}$ . Define

$$\begin{aligned} A_\alpha : M &\longrightarrow M, \\ x &\longmapsto \alpha x \alpha^{-1} - x. \end{aligned}$$

Let  $s_{E/F} : M \rightarrow E$  be a tame corestriction on  $M$ , relative to  $E/F$ . By definition,  $s_{E/F}$  is an  $(E, E)$ -bimodule homomorphism  $M \rightarrow E$  such that  $s_{E/F}(\mathfrak{a}) = \mathfrak{o}_E$ . For integers  $i < j$ , we have exact sequences

$$\begin{aligned} (5.1.1) \quad 0 &\rightarrow \mathfrak{p}_E^i \longrightarrow \mathfrak{p}^i \xrightarrow{A_\alpha} \mathfrak{p}^i \xrightarrow{s_{E/F}} \mathfrak{p}_E^i \rightarrow 0, \\ 0 &\rightarrow \mathfrak{p}_E^i/\mathfrak{p}_E^j \longrightarrow \mathfrak{p}^i/\mathfrak{p}^j \xrightarrow{A_\alpha} \mathfrak{p}^i/\mathfrak{p}^j \xrightarrow{s_{E/F}} \mathfrak{p}_E^i/\mathfrak{p}_E^j \rightarrow 0. \end{aligned}$$

As in 2.1, let  $\mu_F$  be a character of  $F$  of level one and set  $\mu_M = \mu_F \circ \text{tr}_M$ . Let  $w_{E/F}$  denote the wild exponent of the field extension  $E/F$  (1.6.1).

**Lemma.**

(1) *There is a unique character  $\mu_E$  of  $E$ , of level one, so that*

$$(5.1.2) \quad \mu_M(x) = \mu_E(s_{E/F}(x)), \quad x \in M.$$

(2) *There is a unique  $d \in E$ , of valuation  $w_{E/F}$ , such that  $s_{E/F}(y) = yd$ ,  $y \in E$ .*

*Proof.* Part (1) is [15] (1.3.7), part (2) follows from [15] (1.3.8).  $\square$

**5.2.** Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ . Define  $l_E(\theta)$  as the least integer  $l \geq 0$  for which the character  $\theta|U_E^{l+1}$  is trivial.

**Proposition.** *Abbreviate  $w = w_{E/F}$ .*

- (1) *If  $m > 2w$ , then  $l_E(\theta) = m - w$ , for all  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ .*
- (2) *If  $m \leq 2w$ , then  $0 \leq l_E(\theta) \leq m/2$ . If  $l$  is an integer,  $0 \leq l \leq m/2$ , there exists  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  such that  $l_E(\theta) = l$ .*

*Proof.* Let  $y \in E$ ,  $v_E(y) \geq [m/2] + 1$ . The description (2.3.1) of  $\theta$  gives

$$\theta(1+y) = \psi_M * \alpha(1+y) = \mu_E(\alpha s_{E/F}(y)),$$

for a tame corestriction  $s_{E/F}$  and a character  $\mu_E$  of  $E$ , as in 5.1 Lemma. Also,  $v_E(s_{E/F}(y)) = v_E(y) + w$ . Consequently, if  $2w < m$ , the character  $\theta$  is non-trivial on  $U_E^{1+[m/2]}$  and  $l_E(\theta) = m - w$ . Otherwise,  $\theta$  is trivial on  $U_E^{1+[m/2]}$  and assertion (2) follows from the description in 2.3.  $\square$

**Warning.** The variation of  $l_E(\theta)$  with  $E$  is unstable and quite subtle. We explore and exploit this in section 6.

**5.3.** We use the notation  $j_\infty, \bar{j}_\infty$  of (4.2.2). We spend the rest of this section proving:

**Theorem.** *Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M = M_{p^r}(F)$ ,  $r \geq 1$ , in which  $E = F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $p$  does not divide  $m$ . Set  $\varsigma = m/p^r$  and let  $w = w_{E/F}$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  have endo-class  $\Theta$  and suppose that*

$$(5.3.1) \quad l_E(\theta) \leq \max\{0, m - w\}.$$

- (1) If  ${}^2\Psi_{(E/F,\varsigma)}(x)$  has an odd number of jumps, then  $\Psi_\Theta(x) = {}^2\Psi_{(E/F,\varsigma)}(x)$ ,  $0 \leq x \leq \varsigma$ .
- (2) If  $m > 2w$ , then  $l_E(\theta) = m-w$  and  $\Psi_\Theta(x) = {}^2\Psi_{(E/F,\varsigma)}(x)$ ,  $0 \leq x \leq \varsigma$ .
- (3) If  $w$  is divisible by  $p$ , then  $\Psi_\Theta(x) = {}^2\Psi_{(E/F,\varsigma)}(x)$ ,  $0 \leq x \leq \varsigma$ .
- (4) Suppose that  $m > w \geq m/2$ , that  $w$  is not divisible by  $p$ , and that  ${}^2\Psi_{(E/F,\varsigma)}$  has an even number of jumps. There is a unique character  $\phi$  of  $U_E^{m-w}$ , trivial on  $U_E^{1+m-w}$ , with the following property.
  - (a) The relation  $\Psi_\Theta(x) = {}^2\Psi_{(E/F,\varsigma)}(x)$  holds for all  $x$ ,  $0 \leq x \leq \varsigma$ , if and only if  $\theta \mid U_E^{m-w} \neq \phi$ .
  - (b) If  $\theta \mid U_E^{m-w} = \phi$ , then

$$\begin{aligned} \Psi_\Theta(x) &= {}^2\Psi_{(E/F,\varsigma)}(x), & 0 \leq x \leq j_\infty, \quad \bar{j}_\infty \leq x \leq \varsigma, \\ \Psi_\Theta(x) &< {}^2\Psi_{(E/F,\varsigma)}(x), & j_\infty < x < \bar{j}_\infty. \end{aligned}$$

*Remarks.*

- (1) The hypothesis of part (1) holds if and only if  ${}^2\Psi'_{(E/F,\varsigma)}(x) \neq 1$  for  $0 < x < \varsigma$  (4.2 Proposition). It is valid if  $w \geq m(p^r-1)/(p^r+1)$  (4.2 Remark). In particular, if  $w \geq m$  then part (1) applies.
- (2) In part (2), the hypothesis (5.3.1) holds for all  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  (5.2 Proposition). This case therefore subsumes 4.6 Example.
- (3) Regarding (3), the case  $w \equiv 0 \pmod{p}$  can only occur when  $F$  has characteristic 0: see 1.8.
- (4) A form of the character  $\phi$  in part (4) is written down in (5.12.2) below. It may or may not be trivial: see 7.3 Remark. By 4.2 Proposition, we have  ${}^2\Psi'_{(E/F,\varsigma)}(x) = 1$  if and only if  $j_\infty < x < \bar{j}_\infty$ .

The division into cases (1)–(4) is not exclusive. Certainly (3) can overlap either (1) or (2). In certain circumstances, notably when  $p = 2$ , (1) and (2) can overlap. Case (4), however, overlaps no other.

**5.4.** Let  $\mathfrak{p}$  be the Jacobson radical of  $\mathfrak{a}$ . Let  $c \in F$ ,  $v_F(c) = -k$ ,  $k < \varsigma = m/p^r$ . Let  $t < p^r k$  be an integer. We consider formal intertwining between the simple strata  $[\mathfrak{a}, m, t, \alpha]$  and  $[\mathfrak{a}, m, t, \alpha+c]$ . That is, we analyze the congruence

$$(5.4.1) \quad u^{-1}\alpha u \equiv \alpha + c \pmod{\mathfrak{p}^{-t}}, \quad u \in U_{\mathfrak{a}}^1.$$

**Lemma.** *The set of solutions  $u \in U_{\mathfrak{a}}^1$  of (5.4.1) is either empty or constitutes one coset in  $uU_E^1U_{\mathfrak{a}}^{m-t} \in U_E^1U_{\mathfrak{a}}^{m-p^rk}/U_E^1U_{\mathfrak{a}}^{m-t}$ .*

*Proof.* Let  $u \in U_{\mathfrak{a}}^1$  satisfy (5.4.1). Thus  $u$  intertwines the stratum  $[\mathfrak{a}, m, nk, \alpha]$ . If  $v \in U_{\mathfrak{a}}^1$  and  $uv$  satisfies (5.4.1), then  $v$  intertwines the stratum  $[\mathfrak{a}, m, t, \alpha + c]$ . Thus both assertions of the lemma reduce to standard properties [15] (1.5.8).  $\square$

Since  $U_E^1$  commutes with  $\alpha$ , we only ever need consider elements  $u$  of  $U_{\mathfrak{a}}^{m-p^rk}$  satisfying (5.4.1).

**5.5.** We continue with the same notation. In (5.4.1), write  $u = 1+a$ ,  $a \in \mathfrak{p}^{m-p^rk}$ . In this form,

$$(5.5.1) \quad (1+a)^{-1}\alpha(1+a) \equiv \alpha + c \pmod{\mathfrak{p}^{-t}}$$

or, equivalently,

$$(5.5.2) \quad \alpha a - a\alpha \equiv c(1+a) \pmod{\mathfrak{p}^{-t}}.$$

We will use the standard notation  $[x, y] = xy - yx$ , for  $x, y \in M$ .

**Proposition.** *Let  $a \in \mathfrak{p}^{m-p^rk}$  satisfy (5.5.1). If  $y \in E$ ,  $v_E(y) = b \geq 1$ , then*

$$(1+a)(1+y)(1+a)^{-1} \equiv 1 + \bar{y} \pmod{\mathfrak{p}^{b+m-t}},$$

*for an element  $\bar{y} \in E$  such that  $\bar{y} \equiv y \pmod{\mathfrak{p}_E^{b+m-p^rk}}$ .*

*Proof.* We re-arrange the conjugation as

$$(1+a)(1+y)(1+a)^{-1} = 1 + y + [a, y](1+a)^{-1}.$$

Applying the defining relations (5.5.1), (5.5.2), we get

$$\begin{aligned} & [\alpha, [a, y](1+a)^{-1}] \\ &= \alpha[a, y](1+a)^{-1} - [a, y](1+a)^{-1}\alpha \\ &\equiv \alpha[a, y](1+a)^{-1} - [a, y](\alpha + c)(1+a)^{-1} \pmod{\mathfrak{p}^{b+m-p^rk-t}} \\ &\equiv ([\alpha, a]y - y[\alpha, a] - [a, y]c)(1+a)^{-1} \pmod{\mathfrak{p}^{b+m-p^rk-t}} \\ &\equiv (c(1+a)y - yc(1+a) - [a, y]c)(1+a)^{-1} \pmod{\mathfrak{p}^{b-t}} \\ &= 0. \end{aligned}$$

The exact sequences (5.1.1) imply  $[a, y](1+a)^{-1} = v + h$ , for  $v \in \mathfrak{p}_E^{b+m-p^rk}$  and  $h \in \mathfrak{p}^{b+m-t}$ , as required.  $\square$



**5.6.** We continue with the same notation, especially  $\varsigma = m/p^r$  and  $w = w_{E/F}$ .

**Proposition.** *Let  $I$  be an open sub-interval of  $(0, \varsigma)$  on which  $\Psi_{(E/F, \varsigma)}^\times, \Psi_{(E/F, \varsigma)}^+$  are smooth and satisfy*

$$(5.6.1) \quad \Psi_{(E/F, \varsigma)}^\times(x) > \Psi_{(E/F, \varsigma)}^+(x), \quad x \in I.$$

Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ , and suppose

$$(5.6.2) \quad l = l_E(\theta) \leq \max\{0, m-w\}.$$

If  $\theta$  has endo-class  $\Theta$ , then

$$\Psi_\Theta(x) = \Psi_{(E/F, \varsigma)}^\times(x) = {}^2\Psi_{(E/F, \varsigma)}(x), \quad x \in I.$$

*Proof.* By (4.2.1), (4.2.3), we have  $\Psi_{(E/F, \varsigma)}^{\times'}(x) \leq 1 \leq \Psi_{(E/F, \varsigma)}^{+'}(x)$ ,  $0 < x < \varsigma$ . By 4.2 Proposition, the hypothesis (5.6.1) implies that  $\Psi_{(E/F, \varsigma)}^{\times'}(x) < 1$ ,  $x \in I$ . The convexity of  $\psi_{E/F}$  and 1.6 Proposition now imply  $\Psi_{(E/F, \varsigma)}^\times(x) > x - p^{-r}w$ ,  $x \in I$ .

Exactly as in the proof of 2.6 Proposition, the tame lifting properties of 4.4 Proposition and 2.5 Proposition 2 show it is enough to prove the result when  $x$  is an integer. Let  $k$  be an integer,  $k \in I$ , and not a jump of  $\Psi_{(E/F, \varsigma)}^+$ . Let  $(k, c, \chi)$  be a twisting datum (2.5). We apply 5.5 Proposition with  $t = p^r \Psi_{(E/F, \varsigma)}^+(k)$ . By 4.5 Proposition,  $t$  is the least integer for which the congruence (5.5.1) admits a solution  $a \in \mathfrak{p}^{m-p^r k}$ .

Let  $y \in E$  have valuation  $b = 1 + \psi_{E/F}(k)$ . In particular,  $\chi(\det 1+y) = 1$  (cf. 1.3 Proposition). Our hypothesis (5.6.1) amounts to

$$\psi_{K/F}(k) = p^r \Psi_{(E/F, \varsigma)}^\times(k) > p^r \Psi_{(E/F, \varsigma)}^+(k) = t,$$

so  $b > 1+t$ . Thus 5.5 Proposition gives  $\theta^{1+a}(1+y) = \theta(1+\bar{y})$ . Taking first the case  $l = 0$ , we get  $\theta^{1+a}(1+y) = \theta(1+\bar{y}) = 1 = \chi\theta(1+y)$ . In the other case  $0 \leq l \leq m-w$ , we get

$$v_E(\bar{y}-y) \geq b+m-p^r k = 1 + \psi_{E/F}(k) + m - p^r k \geq 1 + m - w,$$

since  $\psi_{E/F}(k) \geq p^r k - w$ . It follows that  $\theta^{1+a}(1+y) = \theta(1+\bar{y}) = \theta(1+y) = \chi\theta(1+y)$ . We conclude in both cases that  $\Psi_\Theta(k) \leq (b-1)/p^r = \Psi_{(E/F, \varsigma)}^\times(k) = {}^2\Psi_{(E/F, \varsigma)}(k)$ .

We go through the same process with  $v_E(y) = b = \psi_{E/F}(k)$ . We choose  $y$  so that  $\chi(\det 1+y) = \chi(N_{E/F} 1+y) \neq 1$ . If  $m > w$ , then

$$v_E(\bar{y}-y) \geq b+m-p^r k > m-w \geq l,$$

whence  $\theta^{1+a}(1+y) = \theta(1+y) \neq \chi\theta(1+y)$ . Thus  $\Psi_\Theta(k) = {}^2\Psi_{(E/F,\varsigma)}(k)$  in this case. If  $m \leq w$  then  $l = 0$  and the same conclusion holds.  $\square$

We prove part (1) of 5.3 Theorem. If we replace the hypothesis (5.6.1) by

$$(5.6.3) \quad \Psi_{(E/F,\varsigma)}^+(x) > \Psi_{(E/F,\varsigma)}^\times(x), \quad x \in I,$$

then, by symmetry, (5.6.1) still holds relative to an interval  $I'$  and  $\Psi_\Theta = {}^2\Psi_{(E/F,\varsigma)} = \Psi_{(E/F,\varsigma)}^+$  on  $I$ . Suppose that  ${}^2\Psi_{(E/F,\varsigma)}$  has an odd number of jumps. The interval  $0 < x < \varsigma$ , with the jumps of  ${}^2\Psi_{(E/F,\varsigma)}$  removed, is covered by a finite union of open intervals on which either (5.6.1) or (5.6.3) holds. The proposition implies  $\Psi_\Theta(x) = {}^2\Psi_{(E/F,\varsigma)}(x)$ , for  $0 < x < \varsigma$ . This proves part (1) of 5.3 Theorem.  $\square$

*Remark.* Suppose that  ${}^2\Psi_{(E/F,\varsigma)}$  has an even number of jumps. Thus  $j_\infty < c_{(E/F,\varsigma)} < \bar{j}_\infty$  (4.2 Lemma). The argument we just used to prove part (1) of the theorem shows equally that  $\Psi_\Theta(x) = {}^2\Psi_{(E/F,\varsigma)}(x)$ , except possibly for  $j_\infty < x < \bar{j}_\infty$ . That is the region where  ${}^2\Psi'_{(E/F,\varsigma)}(x) = 1$  and  $\Psi_{(E/F,\varsigma)}^\times(x) = \Psi_{(E/F,\varsigma)}^+(x) = x-p^{-r}w_{E/F}$ . The remaining part of the proof of 5.3 Theorem therefore concentrates on that interval.

**5.7.** Before picking up the challenge of 5.6 Remark, we need a minor result from elementary linear algebra.

Let  $\mathbf{k}$  be a field and  $V$  a  $\mathbf{k}$ -vector space of finite dimension  $n$ . Let  $\mathbf{n}$  be a regular nilpotent endomorphism of  $V$ . The  $\mathbf{n}$ -stable subspaces of  $V$  are then  $V_j = \mathbf{n}^j(V)$ ,  $0 \leq j \leq n$ .

**Lemma.** *Let  $\mathbf{n}'$  be a nilpotent endomorphism of  $V$  that commutes with  $\mathbf{n}$ . There exists  $a = a(V, \mathbf{n}, \mathbf{n}') \in \mathbf{k}$  such that*

$$\mathbf{n}'(v) \equiv a\mathbf{n}(v) \pmod{V_{j+2}}, \quad v \in V_j,$$

for  $0 \leq j \leq n-2$ . The element  $a$  is non-zero if and only if  $\mathbf{n}'$  is regular.

*Proof.* Let  $\mathbf{m} \in \text{End}_{\mathbf{k}}(V)$  commute with  $\mathbf{n}$ . There is a unique polynomial  $\phi(X) \in \mathbf{k}[X]$ , of degree at most  $n-1$ , such that  $\mathbf{m} = \phi(\mathbf{n})$ . The endomorphism  $\mathbf{m}$  is

nilpotent if and only if  $\phi(0) = 0$ . If this holds, the linear coefficient  $a = \phi'(0)$  satisfies  $\mathfrak{m}(v) \equiv an(v) \pmod{V_{j+2}}$ ,  $v \in V_j$ , as required.  $\square$

We apply the lemma in the following context. Let  $\mathbb{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$  be the residue class field of  $F$ . The endomorphism of the  $p^r$ -dimensional  $\mathbb{k}_F$ -space  $\mathfrak{a}/\mathfrak{p}$ , induced by  $A_\alpha$ , is nilpotent. By (5.1.1), its kernel is the 1-dimensional subspace  $\mathfrak{o}_E/\mathfrak{p}_E$ , so it is a *regular* nilpotent. Let  $V_j = A_\alpha^j(\mathfrak{a}/\mathfrak{p})$ .

**Proposition.** *Let  $r$  be an integer and write  $\zeta = r/m \in \mathfrak{o}_F$ . If  $\beta \in E$  has valuation  $v_E(\beta) = -r$ , then*

$$A_\beta(v) \equiv \zeta A_\alpha(v) \pmod{V_{j+2}},$$

for  $v \in V_j$ ,  $0 \leq j \leq p^r - 2$ .

*Proof.* We may identify  $\mathfrak{a}/\mathfrak{p}$  with  $\mathbb{k}_F^n$ . The conjugation action  $\text{Ad } \alpha : x \mapsto \alpha x \alpha^{-1}$  then permutes the coordinates cyclically with period  $p^r$ , and  $A_\alpha = \text{Ad } \alpha - 1$ . Similarly for  $A_\beta$ , and  $A_\beta + 1 = (A_\alpha + 1)^t$ , for an integer  $t$ ,  $0 \leq t \leq p^r - 1$ , such that  $r \equiv mt \pmod{p^r}$ . The linear term in  $(A_\alpha + 1)^t$  is  $tA_\alpha$ , whence the result follows.  $\square$

**5.8.** We return to the context of 5.6 Remark, and assume that  ${}^2\Psi_{(E/F, \varsigma)}$  has an even number of jumps. Let  $I$  be the (non-empty) open interval  $j_\infty < x < \bar{j}_\infty$ . Thus  $I$  is the largest open set on which  $\Psi_{(E/F, \varsigma)}^\times(x) = \Psi_{(E/F, \varsigma)}^+(x)$ . Indeed, for  $x \in I$ ,

$$\Psi_{(E/F, \varsigma)}^\times(x) = \Psi_{(E/F, \varsigma)}^+(x) = x - p^{-r}w,$$

where  $w = w_{E/F}$ .

Let  $(k, c, \chi)$  be a twisting datum with  $k \in I$ . As in 4.5 Proposition, the least integer  $t$  for which the congruence

$$(5.8.1) \quad (1+a)^{-1}\alpha(1+a) \equiv \alpha + c \pmod{\mathfrak{p}^{-t}}$$

admits a solution  $a \in \mathfrak{p}^{m-p^r k}$  is  $t = p^r k - w$ .

We re-write (5.8.1) in the form

$$(5.8.2) \quad A_\alpha(a) \equiv (1+a)c\alpha^{-1} \pmod{\mathfrak{p}^{m-p^r k+w}},$$

and set

$$(5.8.3) \quad \epsilon = A_\alpha(a) - (1+a)c\alpha^{-1} \in \mathfrak{p}^{m-p^r k+w}.$$

**Lemma.** *The element  $\epsilon$  of (5.8.3) satisfies  $v_E(s_{E/F}(\epsilon)) = m - p^r k + w$ .*

*Proof.* Setting  $t = p^r k - w$ , the congruence

$$(5.8.4) \quad A_\alpha(a') \equiv (1+a')c\alpha^{-1} \pmod{\mathfrak{p}^{1+m-t}}$$

has no solution  $a'$ .

Suppose, for a contradiction, that  $v_E(s_{E/F}(\epsilon)) > m - t$ . Take  $a \in \mathfrak{p}^{m-p^r k}$  satisfying (5.8.2): this determines  $a$  modulo  $\mathfrak{p}_E^{m-p^r k} + \mathfrak{p}^{m-t}$ . Let  $y \in \mathfrak{p}^{m-t}$  and consider the congruence

$$A_\alpha(a+y) \equiv (1+a+y)c\alpha^{-1} \pmod{\mathfrak{p}^{1+m-t}}.$$

Since  $m > p^r k$ , we can neglect the term  $yc\alpha^{-1}$ , so this congruence amounts to

$$A_\alpha(a+y) \equiv (1+a)c\alpha^{-1} \pmod{\mathfrak{p}^{1+m-t}},$$

that is,

$$A_\alpha(y) \equiv -\epsilon \pmod{\mathfrak{p}^{1+m-t}}.$$

We have assumed that  $v_E(s_{E/F}(\epsilon)) > m - t$ , so (5.1.1) gives a solution  $y \in \mathfrak{p}^{m-t}$ . This provides a solution  $a' = a + y$  of (5.8.4), which is impossible.  $\square$

**5.9.** We continue with the notation and assumptions of 5.8. If  $w = w_{E/F} \geq m$ , then  ${}^2\Psi_{(E/F, \varsigma)}$  has an odd number of jumps (4.2 Remark). That is case (1) of 5.3 Theorem and has been done. We therefore assume  $w < m$ .

**Proposition.** *Suppose  $w < m$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  satisfy  $l_E(\theta) \leq m - w$ . Let  $a \in \mathfrak{p}^{m-p^r k}$  be a solution of (5.8.1). Define  $\zeta = w/m \in \mathfrak{o}_F$ . If  $y \in E$  and  $v_E(y) \geq p^r k - w$ , then*

$$(5.9.1) \quad \theta^{1+a}(1+y)/\theta(1+y) = \theta(1-\zeta c\alpha^{-1}y) \mu_M(-\alpha\zeta\epsilon y).$$

*Proof.* Suppose first that  $v_E(y) > p^r k - w$ . The right hand side of (5.9.1) then equals 1. Application of 5.5 Proposition gives the same for the left hand side. To deal with the case  $v_E(y) = p^r k - w$ , we need to strengthen 5.5 Proposition, exploiting our more restrictive hypothesis.

**Lemma 1.** *Let  $y \in E$ ,  $v_E(y) = p^r k - w$ . We have  $(1+a)(1+y)(1+a)^{-1} = 1 + \bar{y} + h$ , for elements  $\bar{y}$  of  $E$  and  $h \in \mathfrak{p}^m$  such that*

$$\begin{aligned}\bar{y} &\equiv y \pmod{\mathfrak{p}_E^{m-p^r k}}, \\ h &\equiv -\zeta \epsilon y \pmod{A_\alpha(\mathfrak{p}^m) + \mathfrak{p}^{m+1}}.\end{aligned}$$

*Proof.* We re-write the defining relation (5.8.1) as  $(1+a)^{-1}\alpha(1+a) = \alpha + c + \delta$ . Thus  $\delta \in \mathfrak{p}^{w-p^r k}$  and

$$\begin{aligned}[\alpha, a] &= (1+a)(c+\delta), \\ A_\alpha(a) &= (1+a)(c+\delta)\alpha^{-1}.\end{aligned}$$

Therefore  $\epsilon = (1+a)\delta\alpha^{-1}$ . We start from the identity

$$(5.9.2) \quad (1+a)(1+y)(1+a)^{-1} = 1 + y + [a, y](1+a)^{-1}$$

and evaluate

$$\begin{aligned}[\alpha, [a, y](1+a)^{-1}] &= \alpha[a, y](1+a)^{-1} - [a, y](1+a)^{-1}\alpha \\ &= \alpha[a, y](1+a)^{-1} - [a, y](\alpha + c + \delta)(1+a)^{-1} \\ &= ([\alpha, a]y - y[\alpha, a] - [a, y](c + \delta))(1+a)^{-1} \\ &= ((1+a)(c + \delta)y - y(1+a)(c + \delta) - [a, y](c + \delta))(1+a)^{-1} \\ &= ((1+a)\delta y - y(1+a)\delta - [a, y]\delta)(1+a)^{-1} \\ &= [\delta, y](1+a)^{-1}.\end{aligned}$$

Substituting for  $\delta$ , we get

$$\begin{aligned}[\delta, y](1+a)^{-1} &= [(1+a)^{-1}\epsilon\alpha, y](1+a)^{-1} \\ &= ((1+a)^{-1}\epsilon\alpha y - y(1+a)^{-1}\epsilon\alpha)(1+a)^{-1} \\ &\equiv [\epsilon\alpha, y] \pmod{\mathfrak{p}}.\end{aligned}$$

Thus

$$A_\alpha((1+a)(1+y)(1+a)^{-1}) \equiv [\epsilon\alpha, y]\alpha^{-1} \equiv [\epsilon, y] \pmod{\mathfrak{p}^{m+1}}.$$

By 5.7 Proposition,

$$[\epsilon, y] = -A_y(\epsilon)y \equiv -\zeta A_\alpha(\epsilon)y \pmod{A_\alpha^2(\mathfrak{p}^m) + \mathfrak{p}^{m+1}},$$

so we may choose  $\bar{y}$ , within the stated conditions, to achieve

$$A_\alpha(h) \equiv -\zeta A_\alpha(\epsilon)y \pmod{A_\alpha^2(\mathfrak{p}^m) + \mathfrak{p}^{m+1}}.$$

Adjusting  $\bar{y}$  by an element of  $\mathfrak{p}_E^m$ , we may therefore take  $h$  to satisfy the desired condition.  $\square$

The elementary identity (5.9.2) implies

$$(1+a)(1+y)(1+a)^{-1} \equiv 1 + y + [a, y] \pmod{\mathfrak{p}^{1+m-w}}.$$

**Lemma 2.** *Let  $v_E(y) = p^r k - w$ . If  $\zeta = w/m \in \mathfrak{o}_F$ , then  $[a, y] \equiv -\zeta A_\alpha(a)y \pmod{\mathfrak{p}^{1+m-w}}$ .*

*Proof.* The defining relation  $A_\alpha(a) \equiv (1+a)c\alpha^{-1} \pmod{\mathfrak{p}^{m-p^r k+w}}$  implies that  $A_\alpha^2(a) \in \mathfrak{p}^{1+m-p^r k}$ . That is,  $A_\alpha(a) \in A_\alpha^{p^r-1}(\mathfrak{p}^{m-p^r k}) + \mathfrak{p}^{1+m-p^r k}$ , whence  $a \in A_\alpha^{p^r-2}(\mathfrak{p}^{m-p^r k}) + \mathfrak{p}^{1+m-p^r k}$ . We apply 5.7 Proposition to get

$$[a, y] = -A_y(a)y \equiv -\zeta A_\alpha(a)y \pmod{A_\alpha^{p^r}(\mathfrak{p}^{m-w}) + \mathfrak{p}^{1+m-w}}.$$

Since  $A_\alpha^{p^r}(\mathfrak{p}^{m-w}) \subset \mathfrak{p}^{1+m-w}$ , we have the result.  $\square$

Lemma 2 implies  $[a, y] = -\zeta A_\alpha(a)y \equiv -\zeta c\alpha^{-1}y \pmod{\mathfrak{p}^{1+m-w}}$ , whence the proposition follows.  $\square$

We assemble a progress report.

**Corollary.** *Let  $I$  be the interval  $j_\infty < x < \bar{j}_\infty$ . If  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  has endo-class  $\Theta$  and  $l_E(\theta) \leq \max\{0, m-w\}$ , then*

$$\begin{aligned} \Psi_\Theta(x) &= {}^2\Psi_{(E/F, \varsigma)}(x), & x \notin I, \\ \Psi_\Theta(x) &\leq {}^2\Psi_{(E/F, \varsigma)}(x), & x \in I. \end{aligned}$$

*Proof.* The first assertion is 5.6 Remark. If  $v_E(y) = 1+p^r k - w$ , (5.9.1) gives

$$\theta^{1+a}(1+y)/\theta(1+y) = 1 = \chi \circ N_{E/F}(1+y),$$

whence follows the second assertion.  $\square$

**5.10.** We prove part (2) of 5.3 Theorem.

**Proposition.** *Suppose  $m > 2w_{E/F}$ . If  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  has endo-class  $\Theta$ , then  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x)$ ,  $0 \leq x \leq \varsigma_\Theta$ .*

*Proof.* In this case,  $l_E(\theta) = m - w > m/2$  (5.2 Proposition) and  $\theta(1+y) = \mu_M(\alpha y)$ , for  $y \in E$ ,  $v_E(y) \geq 1 + [m/2]$  (2.3.1). We may assume that  $\Psi_\Theta$  has an even number of jumps, since the other case has been dealt with in part (1) of the theorem.

Let  $k$  be an integer at which  $\Psi_\Theta$  is smooth and  $\Psi_{(E/F, \varsigma)}^\times(k) = \Psi_{(E/F, \varsigma)}^+(k) = k - p^{-r}w$ . We apply the formula (5.9.1), with  $v_E(y) \geq p^r k - w$ . We get

$$\begin{aligned} \theta(1 - \zeta c \alpha^{-1} y) \mu_M(-\alpha \zeta \epsilon y) &= \theta(1 - \zeta c \alpha^{-1} y) \mu_M(\alpha \zeta c \alpha^{-1} y) \mu_M(\alpha \zeta a c \alpha^{-1} y) \\ &= \mu_M(-\alpha \zeta c \alpha^{-1} y) \mu_M(\alpha \zeta c \alpha^{-1} y) \mu_M(\alpha \zeta a c \alpha^{-1} y) \\ &= \mu_M(ac \zeta y), \quad \text{so} \\ \theta^{1+a}(1+y)/\theta(1+y) &= \mu_M(ac \zeta y). \end{aligned}$$

Here,  $\chi(N_{E/F}(1+y)) = \mu_M(cy)$ , so we need to show that the character

$$(5.10.1) \quad 1+y \mapsto \mu_M((1-\zeta a)cy), \quad y \in \mathfrak{p}_E^{p^r k - w},$$

is non-trivial.

The relation  $A_\alpha(a) \equiv (1+a)c\alpha^{-1} \pmod{\mathfrak{p}^{m-p^r k+w}}$  implies  $v_E(s_{E/F}(a)) \geq w$ . If  $v_E(s_{E/F}(a)) > w$ , (5.10.1) reduces to  $1+y \mapsto \mu_M(cy)$ , which is surely not trivial.

Let  $c$  range over the non-zero elements of  $\mathfrak{p}_F^{-k}/\mathfrak{p}_F^{1-k}$ , let  $(k, c, \chi_c)$  be a twisting datum. Let  $1+a_c$  be the conjugating element corresponding to  $c$  in (5.8.1). If  $v_E(s_{E/F}(a_c)) > w$  for some  $c$ , the character (5.10.1) is non-trivial for that  $c$ . Therefore  $\Psi_\Theta(k) = k - p^{-r}w = {}^2\Psi_{(E/F, \varsigma)}(k)$ . The smoothness hypothesis and 2.5 Proposition 1 imply that this holds for all  $c$ , whence (5.10.1) is non-trivial for all  $c$ . We are so reduced to the case where  $s_{E/F}(a_c)$  has valuation  $w$  for all  $c$ .

We can also vary  $k$ . On the interval under consideration, we have  $\Psi_\Theta(x) \leq {}^2\Psi_{(E/F, \varsigma)}(x) = x - p^{-r}w$  (5.9 Corollary). The convexity of  $\Psi_\Theta$  means it is enough to prove equality at *one* interior point of the interval. The line  $y = x - w/p^r$  crosses the line  $x+y = \varsigma$  where  $p^r x = (m+w)/2$ . By symmetry, we may choose our  $k$  so that  $p^r k < (m+w)/2$ . That is,

$$(5.10.2) \quad m - p^r k > (m-w)/2 > w/2.$$

Let us agree to put  $a_0 = 0$ . Conjugating the defining relation

$$(1+a_c)^{-1}\alpha(1+a_c) \equiv \alpha+c \pmod{\mathfrak{p}^{w-p^r k}}$$

by  $(1+a_{c'})$ , condition (5.10.2) yields

$$\begin{aligned} a_{c+c'} &\equiv a_c + a_{c'} \pmod{\mathfrak{p}^{w+1}}, \\ s_{E/F}(a_{c+c'}) &\equiv s_{E/F}(a_c + a_{c'}) \pmod{\mathfrak{p}_E^{w+1}}, \end{aligned} \quad c, c' \in \mathfrak{p}_F^{-k}/\mathfrak{p}_F^{1-k}.$$

Thus  $c \mapsto s_{E/F}(a_c)$  is a homomorphism  $\mathfrak{p}_F^{-k}/\mathfrak{p}_F^{1-k} \rightarrow \mathfrak{p}_E^w/\mathfrak{p}_E^{1+w}$ . It cannot be surjective, since  $s_{E/F}(1+a_c) \notin \mathfrak{p}_E^{1+w}$  by 5.8 Lemma. It therefore has a non-trivial kernel, contradicting our hypothesis. Consequently,  $s_{E/F}(a_c) \in \mathfrak{p}_E^{1+w}$  for all  $c$ , and we are done.  $\square$

*Remark.* Consider the case  $j_\infty = j_\infty(E|F) < \varsigma/2$  of 4.6 Example. Corollary 1.6 gives  $p^r j_\infty > w$ , so  $2w < m$ . In other words, 4.6 Example is subsumed by the proposition.

**5.11.** We prove part (3) of 5.3 Theorem.

**Proposition.** *Suppose  $m/2 < w < m$  and that  $w \equiv 0 \pmod{p}$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  satisfy  $l_E(\theta) \leq m-w$ . If  $\Theta$  is the endo-class of  $\theta$ , then  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x)$ ,  $0 \leq x \leq \varsigma_\Theta$ .*

*Proof.* In the formula (5.9.1),

$$\theta^{1+a}(1+y)/\theta(1+y) = \theta(1-\zeta c\alpha^{-1}y) \mu_M(-\alpha\zeta\epsilon y),$$

we have  $\zeta \equiv 0 \pmod{\mathfrak{p}_F}$ , so it reduces to

$$\theta^{1+a}(1+y)/\theta(1+y) = 1 \neq \chi \circ N_{E/F}(1+y),$$

for some choice of  $y \in \mathfrak{p}_E^{p^r k-w}$ .  $\square$

**5.12.** We prove part (4) of 5.3 Theorem. In this situation, the element  $\zeta$  of 5.9 Proposition is a unit in  $F$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  have endo-class  $\Theta$  and  $l_E(\theta) \leq \max\{0, m-w\}$ .

**Proposition.** *Suppose that  $m > w \geq m/2$  and  $w \not\equiv 0 \pmod{p}$ . Assume that  ${}^2\Psi_{(E/F, \varsigma)}$  has an even number of jumps, so that  ${}^2\Psi'_{(E/F, \varsigma)}$  takes the value 1 on the non-empty open interval  $I$ ,  $j_\infty < x < \bar{j}_\infty$ .*



There is a unique character  $\phi$  of  $U_E^{m-w}/U_E^{1+m-w}$  with the following property: a character  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ , with  $l_E(\theta) \leq m-w$  and endo-class  $\Theta$ , satisfies  $\Psi_\Theta = \Psi_{(E/F, \varsigma)}$  if and only if  $\theta|_{U_E^{m-w}} \neq \phi$ .

If  $\theta|_{U_E^{m-w}} = \phi$ , then  $\Psi_\Theta(x) < \Psi_{(E/F, \varsigma)}(x)$  for all  $x \in I$ .

*Proof.* We know that  $\Psi_\Theta(x) \leq {}^2\Psi_{(E/F, \varsigma)}(x)$ , with equality outside the (non-empty) interval where  $\Psi_{(E/F, \varsigma)}^\times(x) = \Psi_{(E/F, \varsigma)}^+(x)$ . On that interval,

$$\Psi_{(E/F, \varsigma)}^\times(x) = \Psi_{(E/F, \varsigma)}^+(x) = x - w/p^r.$$

We write out again the formula (5.9.1)

$$(5.12.1) \quad \theta^{1+a_c}(1+y)/\theta(1+y) = \theta(1-\zeta c\alpha^{-1}y) \mu_M(-\alpha\zeta\epsilon y),$$

where  $a_c \in \mathfrak{p}^{m-p^r k}$  satisfies (5.8.1). If this character reduces to  $\chi \circ N_{E/F}(1+y) = \mu_M(cy)$ , then  $\Psi_\Theta(k) < {}^2\Psi_{(E/F, \varsigma)}(k)$ . Otherwise we have equality.

We re-write the last factor in (5.12.1) as

$$\mu_M(-\alpha\zeta\epsilon y) = \mu_M(\zeta cy) \mu_M(\zeta a_c cy)$$

and the first as

$$\theta(1-\zeta c\alpha^{-1}y) = \theta(1 - \alpha^{-1}\zeta cy).$$

We have to work out just when the character of  $U_E^{p^r k-w}/U_E^{1+p^r k-w}$ , given by

$$\Xi_{\theta, c} : 1+y \mapsto \theta(1-\alpha^{-1}\zeta cy) \mu_M(\zeta cy) \mu_M(\zeta a_c cy) \mu_M(-cy)$$

is trivial. Note that this expression depends only on  $s_{E/F}(a_c) \in \mathfrak{p}_E^w$ .

The triviality or otherwise of  $\Xi_{\theta, c}$  determines the value of  $\mathbb{A}(\chi\Theta, \Theta)$ , so it depends only on  $k$ , not  $c$  (2.5 Proposition 1). That is, if we fix  $\theta$ , and there exists  $c \in \mathfrak{p}_F^{-k}/\mathfrak{p}_F^{1-k} \setminus 0$  such that  $\Xi_{\theta, c}$  is trivial, then it is trivial for all  $c$ . Moreover,  $\Xi_{\theta', c}$  is trivial if and only if  $\theta'$  agrees with  $\theta$  on  $U_E^{m-w}$ .

Next fix  $c$  and consider the function

$$1+y \mapsto \mu_M(\zeta cy) \mu_M(\zeta a_c cy) \mu_M(-cy), \quad y \in \mathfrak{p}_E^{p^r k-w}.$$

Since  $s_{E/F}(a_c) \in \mathfrak{p}_E^w$ , this defines a character of  $U_E^{p^r k-w}/U_E^{1+p^r k-w}$ . We may surely find  $\theta$ , with  $l_E(\theta) \leq m-w$ , such that  $\Xi_{\theta, c}$  is trivial for this, and hence all,  $c$ . This reduces us to the first case and we are done.  $\square$

We have dealt with all cases of 5.3 Theorem.  $\square$

**Complement.** The character  $\phi$  of part (4) of the theorem is

$$(5.12.2) \quad \phi(1+y) = \mu_M(\alpha y(1+a_c-\zeta^{-1})), \quad y \in \mathfrak{p}_E^{m-w}.$$

The character  $\phi$  does not depend on  $c \in \mathfrak{p}_F^{-k}/\mathfrak{p}_F^{1-k}$ . In particular, the valuation  $v_E(s_{E/F}(a_c))$  is independent of  $c$ .

## 6. Variation of parameters

In section 5, we fixed the stratum  $[\mathfrak{a}, m, 0, \alpha]$  and calculated  $\Psi_\Theta$  under restrictive conditions on  $l_{F[\alpha]}(\theta)$ , where  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  is a realization of  $\Theta$ . The emphasis now shifts. We fix the set  $\mathcal{C}(\mathfrak{a}, \alpha)$  and investigate some of the effects of varying  $\alpha$ : this prepares the way for the major results of the next section.

**6.1.** Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M = M_{p^r}(F)$ ,  $r \geq 1$ , satisfying the usual conditions:  $F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $m$  is not divisible by  $p$ . Set  $\varsigma = m/p^r$ . Define  $P(\mathfrak{a}, \alpha)$  as the set of  $\beta \in \mathrm{GL}_{p^r}(F)$  for which  $[\mathfrak{a}, m, 0, \beta]$  is a simple stratum such that  $\mathcal{C}(\mathfrak{a}, \beta) = \mathcal{C}(\mathfrak{a}, \alpha)$ . For  $\beta \in P(\mathfrak{a}, \alpha)$ , the stratum  $[\mathfrak{a}, m, 0, \beta]$  has the same properties as  $[\mathfrak{a}, m, 0, \alpha]$ . Also, if  $\beta \in P(\mathfrak{a}, \alpha)$ , then  $P(\mathfrak{a}, \beta) = P(\mathfrak{a}, \alpha)$ .

*Remarks.*

- (1) Let  $[\mathfrak{a}, q, 0, \gamma]$  be a simple stratum in  $M$ . If the set  $\mathcal{C}(\mathfrak{a}, \alpha) \cap \mathcal{C}(\mathfrak{a}, \gamma)$  is non-empty, then  $q = m$  and  $\mathcal{C}(\mathfrak{a}, \alpha) = \mathcal{C}(\mathfrak{a}, \gamma)$  [15] (3.5.8), (3.5.11), whence  $\|\mathcal{C}(\mathfrak{a}, \alpha)\| = \|\mathcal{C}(\mathfrak{a}, \gamma)\|$ .
- (2) Let  $\mathcal{K}_\mathfrak{a}$  be the group of  $x \in \mathrm{GL}_{p^r}(F)$  such that  $x\mathfrak{a}x^{-1} = \mathfrak{a}$ . For  $\beta_1, \beta_2 \in P(\mathfrak{a}, \alpha)$ , say that  $\beta_1 \sim \beta_2$  if  $\beta_1 U_\mathfrak{a}^m$  is  $\mathcal{K}_\mathfrak{a}$ -conjugate to  $\beta_2 U_\mathfrak{a}^m$ . It is shown in [16] that the sets  $P(\mathfrak{a}, \alpha)/\sim$  and  $\|\mathcal{C}(\mathfrak{a}, \alpha)\|$  are in (non-canonical) bijection.

**Proposition.** *Suppose that  $m > 2w_{F[\alpha]/F}$ . If  $\beta \in P(\mathfrak{a}, \alpha)$  then  $w_{F[\beta]/F} = w_{F[\alpha]/F}$ .*

*Proof.* Let  $\mathfrak{p} = \mathrm{rad} \mathfrak{a}$ . Abbreviate  $w = w_{F[\alpha]/F}$ ,  $w' = w_{F[\beta]/F}$ . By hypothesis,  $m-w \geq [\frac{m}{2}] + 1$ . If  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ , 5.2 Proposition says that  $l_{F[\alpha]}(\theta) = m-w$ . That is,  $\theta$  is trivial on  $U_{F[\alpha]}^{m-w+1}$ , but non-trivial on  $U_{F[\alpha]}^{m-w}$ . The map  $A_\alpha$  of 5.1 induces an endomorphism of  $\mathfrak{p}^{m-w}/\mathfrak{p}^{m+1}$  with kernel  $\mathfrak{p}_{F[\alpha]}^{m-w} + \mathfrak{p}^{m+1}/\mathfrak{p}^{m+1}$  (5.1.1) and similarly with  $w$  replaced by  $w-1$ . So, putting it another way,  $\theta$  is trivial on the kernel of  $A_\alpha|_{\mathfrak{p}^{m-w+1}/\mathfrak{p}^{m+1}}$ , but not on the kernel of  $A_\alpha|_{\mathfrak{p}^{m-w}/\mathfrak{p}^{m+1}}$ .

On the group  $U_\mathfrak{a}^{1+[m/2]}$ , the character  $\theta$  restricts to  $\psi_M * \alpha : 1+x \mapsto \mu_M(\alpha x)$  (2.3.1). The same applies relative to  $\beta$ , so  $\beta \equiv \alpha \pmod{\mathfrak{p}^{-[m/2]}}$  or, in multiplicative terms,  $\beta \equiv \alpha \pmod{U_\mathfrak{a}^{m-[m/2]}}$ . This implies

$$A_\alpha(x) \equiv A_\beta(x) \pmod{\mathfrak{p}^{m+1}}, \quad x \in \mathfrak{p}^{1+[m/2]}.$$

The maps  $A_\alpha, A_\beta$  therefore have the same kernels on the groups  $\mathfrak{p}^{m-w+1}/\mathfrak{p}^{m+1}$  and  $\mathfrak{p}^{m-w}/\mathfrak{p}^{m+1}$ . It follows that  $\theta$  is trivial on  $U_{F[\beta]}^{1+m-w}$ , but not trivial on  $U_{F[\beta]}^{m-w}$ . Therefore  $w = w'$ , as required.  $\square$

**Corollary.** *In the context of the proposition, we have*

$$\begin{aligned} {}^2\Psi_{(F[\beta]/F, \varsigma)}(x) &= {}^2\Psi_{(F[\alpha]/F, \varsigma)}(x), \quad 0 \leq x \leq \varsigma, \\ \psi_{F[\beta]/F} &= \psi_{F[\alpha]/F}. \end{aligned}$$

*Proof.* Let  $\Theta \in \|\mathcal{C}(\mathfrak{a}, \alpha)\|$ . Part (2) of 5.3 Theorem gives

$$(6.1.1) \quad \Psi_{\Theta}(x) = {}^2\Psi_{(F[\alpha]/F, \varsigma)}(x) = {}^2\Psi_{(F[\beta]/F, \varsigma)}(x), \quad 0 \leq x \leq \varsigma,$$

whence follows the first assertion.

If  $\Psi$  denotes any of the functions appearing in (6.1.1), define  $c$  by  $x + \Psi(x) = \varsigma$ , so that  $\psi_{F[\alpha]/F}(x) = \psi_{F[\beta]/F}(x)$  for  $0 \leq x \leq c$ . Let  $j_{\infty} = j_{\infty}(\alpha)$  be the last jump of  $\psi_{F[\alpha]/F}$ . If  $j_{\infty} < c$ , all jumps of  $\psi_{F[\alpha]/F}$  appear in the region  $x < c$  and the second assertion would follow in this case. However, 1.6 Corollary gives

$$j_{\infty} \leq \frac{w_{F[\alpha]/F}}{p^{r-1}(p-1)}.$$

If  $p \geq 3$ ,  $j_{\infty} + \Psi_{\Theta}(j_{\infty}) = 2j_{\infty} - p^{-r}w_{F[\alpha]/F} < \varsigma$ . That is,  $j_{\infty} < c$ . If  $p = 2$  and  $j_{\infty}$  is the only jump of  $\psi_{F[\alpha]/F}$ , then  $j_{\infty} = w_{F[\alpha]/F}/(2^r - 1)$  and  $\psi_{F[\alpha]/F}(j_{\infty}) = 2^{-r}j_{\infty}$ . If  $r \geq 2$ , a simple calculation gives  $j_{\infty} < c$ .

If  $p^r = 2$ , the graph of  $\psi_{F[\alpha]/F}$  consists of segments of the lines  $y = x$ ,  $y = 2x - w$  and the same with  $(\beta, w')$  replacing  $(\alpha, w)$ . In this case, the assertion follows from the proposition. We are left with that where  $p = 2$ ,  $r \geq 2$ ,  $j_{\infty} \geq c$  and  $\psi_{F[\alpha]/F}$  has at least two jumps. Let  $j'$  be the penultimate one.

We show that  $j' < c$ . Abbreviate  $a = w/2^r$ , so that  $2^{-r}\psi_{F[\alpha]/F}(x) = x - a$  for  $x \geq j_{\infty}$ , while  $2^{-r}\psi_{F[\alpha]/F}(x) > x - a$  when  $0 \leq x < j_{\infty}$ . By definition,  $2^{-r}\psi_{F[\alpha]/F}(x) \leq x/4$  when  $0 \leq x \leq j'$ . Let  $x = j''$  be the intersection of  $y = x/4$  with  $y = x - a$ , that is,  $j'' = 4a/3$ . We assert that  $j'' > j'$ . For,  $j' \geq j''$  would imply that the line  $y = x - a$  meets  $y = \psi_{F[\alpha]/F}(x)/2^r$  at a point  $k$  such that  $0 \leq k \leq j' < j_{\infty}$ , which is impossible. On the other hand,  $y = x - a$  meets  $x + y = \varsigma$  at  $x = (a + \varsigma)/2$ . Since  $\psi_{F[\alpha]/F}(x)/2^r > x - a$  for  $x < c$ , we have  $(a + \varsigma)/2 \leq c$ . Altogether,

$$j' < j'' = 4a/3 < (a + \varsigma)/2 \leq c,$$

as asserted.

We have  $\psi_{F[\alpha]/F}(x) = \psi_{F[\beta]/F}(x)$  in the region  $0 \leq x < c$ . The same analysis applies with  $\beta$  replacing  $\alpha$ , so  $j'$  is also the penultimate jump of  $\psi_{F[\beta]/F}$ . Let

$\tilde{\psi}(x)$  be the piecewise linear function agreeing with  $\psi_{F[\alpha]/F}(x) = \psi_{F[\beta]/F}(x)$  for  $x < c$  and smooth for  $x > j'$ . In the region  $x \geq 0$ , we then have

$$\begin{aligned}\psi_{F[\alpha]/F}(x) &= \max\{\tilde{\psi}(x), x-2^{-r}w\} \\ &= \max\{\tilde{\psi}(x), x-2^{-r}w'\} = \psi_{F[\beta]/F}(x),\end{aligned}$$

as required.  $\square$

**6.2.** We use the notation from the start of 6.1, except that we write  $E = F[\alpha]$  and assume  $m \leq 2w_{E/F}$ . This case is more complex and interesting. We first investigate the possibility of changing  $\alpha$  to *raise* the exponent  $w_{E/F}$ .

**Proposition.** *Suppose that  $m/2 \leq w_{E/F} < m$  and that  $w_{E/F} \not\equiv 0 \pmod{p}$ . There exists  $\beta \in P(\mathfrak{a}, \alpha)$  such that  $w_{F[\beta]/F} > w_{E/F}$ .*

*Proof.* Let  $\mathfrak{p}$  be the Jacobson radical  $\text{rad } \mathfrak{a}$  of  $\mathfrak{a}$ . Abbreviate  $w = w_{E/F}$  and let  $s_{E/F} : M \rightarrow E$  be a tame corestriction. Define a character  $\mu_E$  of  $E$  by  $\mu_E \circ s_{E/F} = \mu_M$ .

Take  $b \in \mathfrak{p}^{w-m}$ . Thus  $s_{E/F}(b) \in \mathfrak{p}_E^{w-m}$ . We choose  $b$  so that  $s_\alpha(b) \notin \mathfrak{p}_E^{1+w-m}$  and set  $\beta = \alpha - b$ . Following [15] (2.2.3), the stratum  $[\mathfrak{a}, m, 0, \beta]$  is simple and the field extension  $F[\beta]/F$  is totally ramified of degree  $p^r$ . The hypothesis  $m \leq 2w$  implies  $\beta \equiv \alpha \pmod{\mathfrak{p}^{-[m/2]}}$  and, following the discussion in 2.3,  $\beta \in P(\mathfrak{a}, \alpha)$ .

Write  $E' = F[\beta]$ . The relation  $H^1(\beta, \mathfrak{a}) = H^1(\alpha, \mathfrak{a})$  implies that any  $y \in \mathfrak{p}_E^{m-w}$  takes the form

$$(6.2.1) \quad y = y' + h,$$

where  $y' \in \mathfrak{p}_{E'}^{m-w}$  and  $h \in \mathfrak{p}^{1+[m/2]}$  (and symmetrically). The element  $[\beta, y] = [\beta, h] = -[b, y]$  lies in  $\mathfrak{a}$ . By (5.1.1), we may choose the decomposition (6.2.1) so that  $h \in \mathfrak{p}^m$ . That done, 5.7 Proposition gives

$$[\beta, h] = -[b, y] = A_y(b)y \equiv \zeta A_\beta(b)y \pmod{\mathfrak{p} + A_\beta^2(\mathfrak{a})},$$

where  $\zeta = (w-m)/m$ . Note here that, if  $y \in \mathfrak{p}_E^{1+m-w}$ , we may take  $h \in \mathfrak{p}^{m+1}$  and the same relation holds trivially. So, we may further refine (6.2.1) to get

$$(6.2.2) \quad h \equiv \zeta by\beta^{-1} \pmod{\mathfrak{p}^{m+1} + A_\beta(\mathfrak{p}^m)}.$$

In multiplicative terms, the definition of  $\beta$  gives  $\beta \equiv \alpha \pmod{U_\alpha^w}$  and therefore  $\beta^{-1} \equiv \alpha^{-1} \pmod{\mathfrak{p}^{m+w}}$ . It follows that

$$(6.2.3) \quad \begin{aligned} \zeta by\alpha^{-1} &\equiv \zeta by\beta^{-1} \pmod{\mathfrak{p}^{m+1}}, \quad \text{and} \\ A_\alpha(\mathfrak{p}^m) + \mathfrak{p}^{m+1} &= A_\beta(\mathfrak{p}^m) + \mathfrak{p}^{m+1}. \end{aligned}$$

The relation (6.2.1) gives  $\mu_M(\beta y) = \mu_M(\beta y') \mu_M(\beta h)$ , while (6.2.2), (6.2.3) yield  $\mu_M(\beta h) = \mu_M(\zeta \beta y)$ . On the other hand,  $\mu_M(\beta y) = \mu_E(s_{E/F}(\alpha - b)y)$  by definition, so

$$\mu_M(\beta y') = \mu_M((\alpha - (\zeta + 1)b)y)$$

The assumption  $w \not\equiv 0 \pmod{p}$  implies  $\zeta \not\equiv -1 \pmod{\mathfrak{p}_F}$ . As  $v_E(s_{E/F}(\alpha)) = w - m$ , we may choose the original element  $b$  so that

$$(6.2.4) \quad (\zeta + 1)s_{E/F}(b) \equiv s_{E/F}(\alpha) \pmod{\mathfrak{p}_E^{1+w-m}}.$$

This gives  $\mu_M(\beta y') = 1$  for all  $y' \in \mathfrak{p}_{E'}^{m-w}$ , whence  $w_{E'/F} > w$ .  $\square$

**Corollary.** *Suppose that  $m \leq 2w_{E/F}$ . There exists  $\beta \in P(\mathfrak{a}, \alpha)$  with the following property. If  $E' = F[\beta]$ , then either*

- (1)  $w_{E'/F} \geq w_{E/F}$  and  $w_{E'/F} \equiv 0 \pmod{p}$ , or
- (2)  $w_{E'/F} \geq m$ .

*The element  $\beta$  may be chosen so that  $b = \alpha - \beta$  lies in  $\mathfrak{p}^{w_{E/F}-m}$  and satisfies (6.2.4).*

*Proof.* If  $w_{E/F}$  is divisible by  $p$ , there is nothing to do. Otherwise, we construct  $E_1 = F[\beta]$  following the proposition. If either  $w_{E_1/F} \geq m$  or  $w_{E_1/F} \equiv 0 \pmod{p}$ , there is nothing more to do. So, assume that  $w_{E_1/F} < m$  or  $w_{E_1/F} \not\equiv 0 \pmod{p}$ . Set  $w_1 = w_{E_1/F}$ . Following the procedure as before, we construct an element  $\gamma \equiv \beta \pmod{\mathfrak{p}^{m-w_1}}$  such that  $w_{F[\gamma]/F} > w_1$ . The congruence condition on  $\gamma$  ensures that  $b_1 = \alpha - \gamma$  satisfies (6.2.4). We iterate this procedure as necessary until we achieve either (1) or (2).  $\square$

*Remark.* When case (2) of the corollary applies, the function  ${}^2\Psi_{(E'/F, m/p^r)}$  has an odd number of jumps (4.2 Proposition).

**6.3.** We retain the notation of 6.2, in particular  $E = F[\alpha]$  and  $w = w_{E/F}$ . The element  $\beta$  constructed in the proof of 6.2 Proposition has useful properties relative to simple characters.

**Proposition.** *Let  $\beta = \alpha - b \in P(\mathfrak{a}, \alpha)$ , where  $b \in \mathfrak{p}^{w-m}$  satisfies (6.2.4).*

- (1) *If  $\xi \in \mathcal{C}(\mathfrak{a}, \alpha)$  satisfies  $\xi(1+y) = \mu_M(\alpha y)$ ,  $y \in \mathfrak{p}_E^{m-w}$ , then*

$$l_{F[\beta]}(\xi) = l_E(\xi) = m - w.$$

- (2) *If  $\xi \in \mathcal{C}(\mathfrak{a}, \alpha)$  satisfies  $l_E(\xi) > m - w$ , then  $l_{F[\beta]}(\xi) = l_E(\xi)$ .*

One may choose the element  $\beta = \alpha - b$  to satisfy the requirements of 6.2 Corollary.

*Proof.* For part (1), we return to the proof of 6.2 Proposition and pick it up after (6.2.3). We evaluate

$$\xi(1+y') = \mu_M(\alpha y) \mu_M(\alpha h) = \mu_M((\alpha - \zeta b)y).$$

As  $\zeta s_{E/F}(b) \equiv \zeta s_{E/F}(\alpha)(\zeta + 1)^{-1} \pmod{\mathfrak{p}_E^{1+w-m}}$ , so by (6.2.4)

$$\xi(1+y') = \mu_E((\zeta + 1)^{-1} s_{E/F}(\alpha)y), \quad y' \in \mathfrak{p}_{F[\beta]}^{m-w},$$

and, since  $1 + \zeta \not\equiv 0 \pmod{p}$ , part (1) of the proposition follows.

In part (2), let  $l = l_E(\xi)$ . Let  $y \in \mathfrak{p}_E^l$  and write  $y = y' + h$ , with  $y' \in \mathfrak{p}_{F[\beta]}^l$  and  $h \in \mathfrak{p}^{m+1}$ . For suitable choice of  $y$ , we have  $\xi(1+y') = \xi(1+y) \neq 1$ . Likewise, if  $y \in \mathfrak{p}_E^{1+l}$ , we get  $\xi(1+y') = \xi(1+y) = 1$ .

The final assertion is clear.  $\square$

**6.4.** We give a complementary result concerning *lowering* of the different, in the case  $2w_{E/F} > m$ .

**Theorem.** Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M = M_{p^r}(F)$  in which  $E = F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $p$  does not divide  $m$ . Suppose  $m < 2w_{E/F}$ . Let  $d$  be an integer such that

$$(6.4.1) \quad \begin{aligned} 1 \leq d \leq m/2, \quad d > \max\{0, m - w_{E/F}\}, \\ d \not\equiv m \pmod{p}. \end{aligned}$$

There exists  $\beta \in P(\mathfrak{a}, \alpha)$  such that

$$(6.4.2) \quad w_{E'/F} = m - d < w_{E/F}, \quad E' = F[\beta].$$

Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  and write  $l = l_E(\theta)$ .

(1) If  $l < d$ , then

(a)  $l_{E'}(\theta) = d$  if  $d \not\equiv 0 \pmod{p}$ , or

(b)  $l_{E'}(\theta) < d$  if  $d \equiv 0 \pmod{p}$ .

(2) If  $l > d$ , then  $l_{E'}(\theta) = l$ .

(3) Suppose  $l = d$ .

(a) If  $d \not\equiv 0 \pmod{p}$ , then  $l_{E'}(\theta) \leq d$ , with both equality and inequality occurring.

(b) If  $d \equiv 0 \pmod{p}$ , then  $l_{E'}(\theta) = d$ .

*Proof.* Writing  $\mathfrak{p} = \text{rad } \mathfrak{a}$ , let  $b \in \mathfrak{p}^{-d}$  satisfy  $v_E(s_{E/F}(b)) = -d$ . Arguing as in 6.2, the element  $\beta = \alpha + b$  lies in  $P(\mathfrak{a}, \alpha)$ . Put  $E' = F[\beta]$ .

**Lemma.** *Let  $y \in \mathfrak{p}_E^d$ . There exist  $y' \in \mathfrak{p}_{E'}^d$  and  $h \in \mathfrak{p}^m$  such that  $y = y' + h$ . If  $y \in \mathfrak{p}_E^{d+1}$ , then  $y' \in \mathfrak{p}_{E'}^{d+1}$  and  $h \in \mathfrak{p}^{m+1}$ .*

*Proof.* The relation  $H^1(\beta, \mathfrak{a}) = H^1(\alpha, \mathfrak{a})$  implies  $y = y' + h$ , with  $y' \in \mathfrak{p}_{E'}^d$  and  $h \in \mathfrak{p}^{1+[m/2]}$ . Thus  $[\beta, h] = [\beta, y] = [b, y]$ . Surely  $[b, y] \in \mathfrak{a}$ . Since  $\beta$  is minimal over  $F$ , we may adjust the choice of  $y'$  to achieve  $h \in \mathfrak{p}^m$  (5.1.1). This gives the first assertion. The second follows similarly.  $\square$

We first show that  $\mu_M(\beta z) = 1$ , for all  $z \in \mathfrak{p}_{E'}^{1+d}$ . The argument of the lemma is symmetrical, so there exist  $y \in \mathfrak{p}_E^{1+d}$  and  $h \in \mathfrak{p}^{m+1}$  such that  $y = z + h$ . The condition  $d > m - w_{E/F}$  implies  $\mu_M(\alpha y) = 1$ . As  $by \in \mathfrak{p}$ , so  $\mu_M(by) = 1$ . Altogether,  $\mu_M(\beta y) = \mu_M(\alpha y)\mu_M(by) = 1$ . Therefore  $1 = \mu_M(\beta z)\mu_M(\beta h) = \mu_M(\beta z)$ , as asserted.

Now take  $z \in E'$  with  $v_{E'}(z) = d$ . Thus  $z = y - h$ , where  $y \in E$  satisfies  $v_E(y) = d$  and  $h \in \mathfrak{p}^m$ . Consequently,  $[\beta, h] = [\beta, y] = [b, y]$ . Setting  $\zeta = -d/m$ , 5.7 Proposition gives

$$[b, y] = -A_y(b)y \equiv -\zeta A_\alpha(b)y \pmod{A_\alpha^2(\mathfrak{a}) + \mathfrak{p}}.$$

Since  $\alpha \equiv \beta \pmod{U_\alpha^{m-d}}$ , we have

$$A_\alpha(a) \equiv A_\beta(a) \pmod{\mathfrak{p}^{a+m-d}}, \quad a \in \mathfrak{p}^a,$$

for any integer  $a$ . So,

$$[\beta, h] = [b, y] \equiv -\zeta A_\beta(b)y \pmod{A_\beta^2(\mathfrak{a}) + \mathfrak{p}}.$$

We may therefore choose the decomposition  $y = z + h$  so that

$$(6.4.3) \quad h \equiv -\zeta by\beta^{-1} \equiv -\zeta by\alpha^{-1} \pmod{A_\beta(\mathfrak{p}^m) + \mathfrak{p}^{m+1}}.$$

We apply the character  $\mu_M * \beta$  to the relation  $y = z + h$ . Since  $\mu_M(\alpha y) = 1$  (because  $d > m - w$ ), we get

$$\begin{aligned} \mu_M(by) &= \mu_M(\beta y) = \mu_M(\beta z)\mu_M(\beta h) \\ &= \mu_M(\beta z)\mu_M(\alpha h) \\ &= \mu_M(\beta z)\mu_M(-\zeta by), \end{aligned}$$

whence  $\mu_M((1+\zeta)by) = \mu_M(\beta z)$ . Our hypothesis  $d \not\equiv m \pmod{p}$  implies that  $\zeta \not\equiv -1 \pmod{p}$  so, for some choice of  $z$ , we get  $\mu_M(\beta z) \neq 1$ . This proves the first assertion (6.4.2) of the theorem.

Set  $w' = w_{E'/F} = m-d$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha) = \mathcal{C}(\mathfrak{a}, \beta)$  and suppose  $l = l_E(\theta) < d = m-w'$ . We calculate the  $E'$ -level  $l_{E'}(\theta)$ . If  $y \in E$ ,  $v_E(y) = 1+d$ , we write  $y = z+h$  as above, with  $z \in E'$  of valuation  $1+d$  and  $h \in A_\alpha(\mathfrak{p}^m) + \mathfrak{p}^{m+1}$ . This gives  $1 = \theta(1+y) = \theta(1+z)\theta(1+h) = \theta(1+z)$ . Thus  $l_{E'}(\theta) \leq d$ . Now take  $y \in E$  of valuation  $d$  and write  $y = z+h$ , where  $v_{E'}(z) = d$  and  $h \in \mathfrak{p}^m$ . Indeed, we may take  $h \equiv -\zeta by \alpha^{-1} \pmod{A_\alpha(\mathfrak{p}^m) + \mathfrak{p}^{m+1}}$  as before. This gives

$$1 = \theta(1+y) = \theta(1+z)\mu_M(\alpha h),$$

and

$$\mu_M(\alpha h) = \mu_M(-\zeta by) = \mu_E(-\zeta y s_\alpha(b)).$$

Suppose  $d \not\equiv 0 \pmod{p}$ . Thus  $\zeta \not\equiv 0 \pmod{\mathfrak{p}_F}$  and we may choose  $y \in \mathfrak{p}_E^d$  so that  $\mu_E(-\zeta y s_\alpha(b)) \neq 1$ . Thus  $\theta(1+z) \neq 1$ , whence  $l_{E'}(\theta) = d$  as required for (1)(a). If  $d \equiv 0 \pmod{p}$ , then  $\zeta \equiv 0 \pmod{\mathfrak{p}_F}$  and  $\theta(1+z) = 1$ . Thus  $l_{E'}(\theta) < d$ , as required for (1)(b).

Part (2) follows from a similar, but easier, argument.

Part (3) is given by a counting argument as follows. Let  $q$  be the cardinality of the residue field  $\mathfrak{o}_F/\mathfrak{p}_F$ . For an integer  $k \leq [m/2]$ , let  $\mathcal{C}(\alpha; \leq k)$  be the set of  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  such that  $l_{F[\alpha]}(\theta) \leq k$ . We use the obvious variations. Note that  $\mathcal{C}(\alpha; \leq k)$  has exactly  $q^k$  elements while  $\mathcal{C}(\alpha; > k)$  has  $q^{[m/2]-k}$  elements.

Part (2) of the theorem gives  $\mathcal{C}(\alpha; > d) \subset \mathcal{C}(\beta; > d)$ , hence  $\mathcal{C}(\alpha; > d) = \mathcal{C}(\beta; > d)$  and  $\mathcal{C}(\alpha; \leq d) = \mathcal{C}(\beta; \leq d)$ . Assertions (3)(a) and (3)(b) now follow from (1)(a) and (1)(b) respectively.  $\square$

We refine the final step of the argument.

**Corollary 1.** *There is a unique character  $\xi$  of  $U_{E'}^d/U_{E'}^{1+d}$  with the following property. If  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  has  $l_{E'}(\theta) = d$ , then  $l_E(\theta) < d$  if and only if  $\theta|U_{E'}^d = \xi$ . The character  $\xi$  is trivial if and only if  $d \equiv 0 \pmod{p}$ .*

*Proof.* Let  $\theta_0 \in \mathcal{C}(\mathfrak{a}, \alpha)$  have  $l_E(\theta_0) = 0$  and endo-class  $\Theta_0$ . Let  $\xi$  be the restriction of  $\theta_0$  to  $U_{E'}^d$ . By assertion (1) of the theorem, this character  $\xi$  is trivial if and only if  $d \equiv 0 \pmod{p}$ . Let  $\theta' \in \mathcal{C}(\mathfrak{a}, \alpha)$  have endo-class  $\Theta'$ . If  $\mathbb{A}$  is the canonical ultrametric on  $\mathcal{E}(F)$ , then  $l_E(\theta') < d$  if and only if  $\mathbb{A}(\Theta_0, \Theta') < p^{-r}d$ . This condition is also equivalent to  $\theta'$  agreeing with  $\theta_0$  on  $U_{E'}^d$ .  $\square$

Remark the parallel with 5.3 Theorem (4).



**Corollary 2.** *Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  satisfy  $l_E(\theta) = d$ . In the theorem, one may choose  $\beta$  so that  $l_{E'}(\theta) = d$ .*

*Proof.* If  $d \equiv 0 \pmod{p}$ , there is nothing to do, so we assume the contrary. Let  $y \in \mathfrak{p}_E^d$ , and write  $y = z + h$ , for  $z \in \mathfrak{p}_{E'}^d$  and  $h \in \mathfrak{p}^m$ , satisfying (6.4.3). Thus  $\theta(1+y) = \theta(1+z) \mu_M(\alpha h) = \theta(1+z) \mu_M(-\zeta by)$ . The function  $1+y \mapsto \mu_M(-\zeta by)$  represents a non-trivial character of  $U_E^d/U_E^{1+d}$ . We may choose  $b$  at the beginning so that  $\mu_M(-\zeta by) \neq \theta(1+y)$ , for some  $y \in \mathfrak{p}_E^d$ . This gives  $\theta(1+z) \neq 1$  and  $l_{E'}(\theta) = d$ , as required.  $\square$

## 7. The Herbrand function

In this section, we state and prove the main results concerning the Herbrand function  $\Psi_\Theta$ ,  $\Theta \in \mathcal{E}^C(F)$ .

**7.1.** We introduce some new terminology.

**Definition.** Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  be a realization of  $\Theta$ , on a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M = M_{p^r}(F)$ . Let  $E = F[\alpha]$  and  $l = l_E(\theta)$ . Say that  $\alpha$  is  $\theta$ -conformal if

$$\theta(1+y) = \mu_M(\alpha y), \quad y \in \mathfrak{p}_E^{1+[l/2]}.$$

Say  $\alpha$  is *weakly  $\theta$ -conformal* if

$$\theta(1+y) = \mu_M(\alpha y), \quad y \in \mathfrak{p}_E^l.$$

In this situation, we might equally say that  $\theta$  is  $\alpha$ -conformal. Let  $\mathcal{C}^*(\mathfrak{a}, \alpha)$  be the set of  $\alpha$ -conformal  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ . Surely  $\mathcal{C}^*(\mathfrak{a}, \alpha)$  is not empty.

Let  $\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$  be the set of endo-classes of elements of  $\mathcal{C}^*(\mathfrak{a}, \alpha)$ . The canonical map  $\mathcal{C}^*(\mathfrak{a}, \alpha) \rightarrow \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$  are in canonical bijection (as follows from 2.3 Remark (2)).

**Proposition.** *Let  $\Theta \in \mathcal{E}^C(F)$  be of degree  $p^r$ . The endo-class  $\Theta$  then has a realization  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ , on a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M = M_{p^r}(F)$ , such that  $\alpha$  is  $\theta$ -conformal. That is,  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ .*

*Proof.* Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  be a realization of  $\Theta$  and let  $\mathfrak{p} = \text{rad } \mathfrak{a}$ . Let  $\nu_\theta(\alpha)$  be the least integer  $\nu$  for  $\theta(1+y) = \mu_M(\alpha y)$ ,  $y \in \mathfrak{p}_E^{1+\nu}$ . Certainly  $\nu \leq [m/2]$  (2.3.1). Write  $E = F[\alpha]$  and  $d_\alpha = m - w_{E/F}$ . We have  $\nu \geq [d_\alpha/2]$  since, otherwise, the function

$$1+x \mapsto \mu_M(\alpha x), \quad x \in \mathfrak{p}_E^{\nu+1},$$

is not a character of  $U_E^{1+\nu}$ . If  $\nu_\theta(\alpha) = [d_\alpha/2]$ , there is nothing more to do.

**Lemma.** *Set  $\nu = \nu_\theta(\alpha)$ , and assume that  $\nu > [d_\alpha/2]$ . There exists  $\beta \in P(\mathfrak{a}, \alpha)$  such that  $\beta \equiv \alpha \pmod{\mathfrak{p}^{-\nu}}$  and  $\nu_\theta(\beta) \leq \nu - 1$ . This condition determines the stratum  $[\mathfrak{a}, m, \nu - 1, \beta]$  uniquely, up to formal intertwining.*

*Proof.* Recall that  $\nu \leq [m/2]$ . By hypothesis, the function

$$\xi(1+x) = \theta(1+x) \mu_M(-\alpha x), \quad x \in \mathfrak{p}_E^\nu,$$

represents a non-trivial character of  $U_E^\nu$ , trivial on  $U_E^{1+\nu}$ . Consequently, there exists  $z \in \mathfrak{p}^{-\nu}$  so that

$$\xi(1+x) = \mu_M(zx), \quad x \in \mathfrak{p}_E^\nu.$$

If we choose a tame corestriction  $s_{E/F} : M \rightarrow E$ , this expression depends only on  $s_{E/F}(z)$  and, as  $\xi$  is non-trivial on  $U_E^\nu$ , so  $v_E(s_{E/F}(z)) = -\nu$ . The element  $\beta = \alpha + z$  lies in  $P(\mathfrak{a}, \alpha)$ . Moreover,  $s_{E/F}(z)$  is uniquely determined modulo  $\mathfrak{p}_E^{1-\nu}$ , so the simple stratum  $[\mathfrak{a}, m, \nu - 1, \beta]$  is uniquely determined up to formal intertwining [15] (2.2.8). Set  $L = F[\beta]$ . We show that

$$\theta(1+x) = \mu_M(\beta x), \quad x \in \mathfrak{p}_L^\nu,$$

whence  $\nu_\theta(\beta) \leq \nu - 1$ , as required to prove the lemma.

Since  $x \in L = F[\beta]$ , there is a polynomial  $f(T) \in F[T]$ , of degree at most  $p^r - 1$ , such that  $x = f(\beta)$ . Write

$$f(T) = a_0 + a_1 T + \dots + a_{p^r-1} T^{p^r-1}.$$

The  $L$ -valuations of the terms  $a_i \beta^i$ ,  $0 \leq i \leq p^r - 1$ , are distinct. The condition  $v_L(x) = \nu$  translates as  $\nu \leq p^r v_F(a_i) - mi$  for all  $i$ , with equality for exactly one value of  $i$ . So, if we put  $y = f(\alpha)$ , we get  $v_E(y) = \nu$ . Consider the element

$$t = x - y = f(\beta) - f(\alpha) = \sum_{1 \leq i < p^r} a_i ((\alpha + z)^i - \alpha^i).$$

Expand  $((\alpha + z)^i - \alpha^i)$ . Any fractional  $\mathfrak{a}$ -ideal  $\mathfrak{p}^k$ ,  $k \in \mathbb{Z}$ , is stable under conjugation by  $\alpha$ , so every term in the expansion of  $(\alpha + z)^i - \alpha^i$  lies in  $\alpha^{i-1} z \mathfrak{a} = \mathfrak{p}^{(1-i)m-\nu}$ . Since  $p^r v_F(a_i) \geq mi + \nu$ , the term  $a_i ((\alpha + z)^i - \alpha^i)$  lies in  $\mathfrak{p}^m$ , whence  $t = f(\beta) - f(\alpha) = x - y \in \mathfrak{p}^m$ .

With this element  $t$ , and setting

$$u = (1+t)^{-1} (1+y)^{-1} y t,$$

we have

$$1+x = (1+y)(1+t)(1-u).$$

We use this expression to evaluate  $\theta(1+x)$ . Our choice of  $z$  gives  $\theta(1+y) = \mu_M(\beta y)$  and, since  $t \in \mathfrak{p}^m$ , we have  $\theta(1+t) = \mu_M(\alpha t)$ . As  $yt \in \mathfrak{p}^{m+1}$ , so  $\theta(1-u) = 1$ . This gives

$$\theta(1+x) = \theta(1+y)\theta(1+t) = \mu_M(\beta y)\mu_M(\alpha t).$$

On the other hand,  $zt \in \mathfrak{p}^{m-\nu}$  and  $m-\nu = (m-2\nu)+\nu \geq 1$ , whence  $\mu_M(zt) = 1$ . Altogether,

$$\mu_M(\beta x) = \mu_M(\beta y)\mu_M(\alpha t)\mu_M(zt) = \theta(1+y)\theta(1+t) = \theta(1+x),$$

as required.  $\square$

The proposition now follows.  $\square$

Note that, while the proposition is an existence statement, the proof is constructive.

**7.2.** For the purpose of stating our main results, it is convenient to have a temporary terminology reflecting the structure of 5.3 Theorem. We consider a datum  $(E/F, m)$  in which  $E/F$  is a totally ramified field extension of degree  $p^r$ ,  $r \geq 1$ , and  $m$  is a positive integer not divisible by  $p$ .

**Definition.** Say that  $(E/F, m)$  is *standard* if at least one of the following conditions holds:

- (a)  $m > 2w_{E/F}$ ;
- (b)  $m \leq 2w_{E/F}$  and  ${}^2\Psi_{(E/F, m/p^r)}$  has an odd number of jumps;
- (c)  $m \leq 2w_{E/F}$ ,  ${}^2\Psi_{(E/F, m/p^r)}$  has an even number of jumps and  $w_{E/F} \equiv 0 \pmod{p}$ .

**Lemma.** Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . There is a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M_{p^r}(F)$  such that

- (1)  $\mathcal{C}(\mathfrak{a}, \alpha)$  contains a character  $\theta$  of endo-class  $\Theta$ , and
- (2) the datum  $(F[\alpha]/F, m)$  is standard.

*Proof.* Choose a simple stratum  $[\mathfrak{a}, m, 0, \beta]$  in  $M_{p^r}(F)$  such that  $\Theta \in \|\mathcal{C}(\mathfrak{a}, \alpha)\|$ . If  $m > 2w_{F[\beta]/F}$ , there is nothing to do. Otherwise, we apply 6.2 Corollary.  $\square$

We now state our main results in the order in which we prove them.

**Theorem 1.** *Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  be a realization of  $\Theta$  on a simple stratum  $[\mathfrak{a}, m, 0, \alpha]$  in  $M_{p^r}(F)$  for which the datum  $(F[\alpha]/F, m)$  is standard. Write  $E = F[\alpha]$ ,  $l = l_E(\theta)$  and  $\varsigma = m/p^r = \varsigma_\Theta$ . For any such realization, the following hold.*

- (1) *If  $l \leq \max(0, m - w_{E/F})$ , then  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, \varsigma)}(x)$ ,  $0 \leq x \leq \varsigma$ .*
- (2) *If  $l > \max(0, m - w_{E/F})$  and  $l \not\equiv m \pmod{p}$ , then*

$$(7.2.1) \quad \Psi_\Theta(x) = \max({}^2\Psi_{(E/F, \varsigma)}(x), x - p^{-r}(m-l)), \quad 0 \leq x \leq \varsigma.$$

- (3) *In part (2), the class  $\Theta$  admits a parameter field  $E'/F$  as follows:*

- (i)  *$E' = F[\beta]$ , where  $\beta \in P(\mathfrak{a}, \alpha)$  and  $\beta \equiv \alpha \pmod{\mathfrak{p}^{-l}}$ ;*
- (ii)  *$w_{E'/F} = m-l$  and  $l_{E'}(\theta) = l$ .*

*For any such  $\beta$ ,  $\Psi_\Theta(x) = {}^2\Psi_{(E'/F, \varsigma)}(x)$ ,  $0 \leq x \leq \varsigma$ .*

We shall see in the course of the proof that (7.2.1) also holds in the situation of part (1), but says nothing new there.

**Theorem 2.** *Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M = M_{p^r}(F)$  such that  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ . For any such realization,  $l_{F[\alpha]}(\theta) = \max(0, m - w_{F[\alpha]/F})$  and*

$$(7.2.2) \quad \Psi_\Theta(x) = {}^2\Psi_{(F[\alpha]/F, \varsigma_\Theta)}(x), \quad 0 \leq x \leq \varsigma_\Theta.$$

*Comment.* The endo-class  $\Theta$  has a realization of the required form, by 7.1 Proposition. When proving Theorem 2, we show that (7.2.2) holds provided only that  $\Theta$  has a realization  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  such that  $\alpha$  is *weakly  $\theta$ -conformal*. We will not use that version in the rest of the paper.

**Theorem 3.** *Let  $\Theta \in \mathcal{E}^C(F)$  have degree  $p^r$ . The endo-class  $\Theta$  admits a realization  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  for which  $(F[\alpha]/F, p^r \varsigma_\Theta)$  is standard and also, if  $E = F[\alpha]$ , then either*

- (a)  *$l_E(\theta) \not\equiv m \pmod{p}$ , or*
- (b)  *$w_{E/F} \equiv 0 \pmod{p}$  and  $l_E(\theta) \leq m - w_{E/F}$ .*

Before embarking on the proofs of the theorems, we give a consequence of Theorem 2.

**Corollary.** *Let  $E/F$  be a totally ramified field extension of degree  $p^r$ , and let  $m$  be a positive integer not divisible by  $p$ . There exists  $\Theta \in \mathfrak{E}^C(F)$ , of degree  $p^r$ , such that  $\Psi_\Theta(x) = {}^2\Psi_{(E/F, m/p^r)}(x)$ ,  $0 \leq x \leq \varsigma_\Theta = m/p^r$ .*

*Proof.* View  $E$  as a subfield of  $M = M_{p^r}(F)$  and take  $\alpha \in E$  such that  $v_E(\alpha) = -m$ . There is a unique hereditary  $\mathfrak{o}_F$ -order  $\mathfrak{a}$  in  $M$  such that  $[\mathfrak{a}, m, 0, \alpha]$  is a simple stratum in  $M$ . By Theorem 2, any  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$  has the required property.  $\square$

**7.3.** We prove 7.2 Theorem 1. The existence of a realization with the required properties is given by 7.2 Lemma.

If  $m > 2w_{E/F}$ , then  $l = m - w_{E/F}$ . In this case, part (1) of the theorem is part (2) of 5.3 Theorem. We therefore assume henceforth that  $m \leq 2w_{E/F}$ . The remaining cases of part (1) follow from parts (1) and (3) of 5.3 Theorem.

In part (2), 6.4 Corollary 2 gives an element  $\beta \in P(\mathfrak{a}, \alpha)$  such that  $w_{F[\beta]/F} = m - l$  and  $l_{F[\beta]}(\theta) = l$ . In particular,  $w_{F[\beta]/F} < w_{E/F}$ . Let  $\theta_0$  be the unique element of  $\mathcal{C}(\mathfrak{a}, \alpha)$  with  $l_E(\theta_0) = 0$ . Let  $\Theta_0$  be the endo-class of  $\theta_0$ . From 6.4 Theorem we get  $l_{F[\beta]}(\theta_0) \leq l$ . If the function  ${}^2\Psi_{(F[\beta]/F, \varsigma)}$  has an odd number of jumps, then  ${}^2\Psi_{(F[\beta]/F, \varsigma)} = \Psi_{\Theta_0} = \Psi_\Theta$ , by part (1) of 5.3 Theorem. Moreover,  $x - p^{-r}w_{F[\beta]/F} \leq {}^2\Psi_{(F[\beta]/F, \varsigma)}(x)$  for  $0 \leq x \leq \varsigma$ , so we are done in this case. This also justifies the comment following the statement of the theorem.

Assume therefore that  ${}^2\Psi_{(F[\beta]/F, \varsigma)}$  has an even number of jumps. We have  ${}^2\Psi_{(E/F, \varsigma)}(x) = \Psi_{\Theta_0}(x) \leq {}^2\Psi_{(F[\beta]/F, \varsigma)}(x)$ , by 5.3 Theorem (4). Since  $w_{F[\beta]/F} < w_{E/F}$ , we conclude that  ${}^2\Psi_{(E/F, \varsigma)}(x) = \Psi_{\Theta_0}(x) < {}^2\Psi_{(F[\beta]/F, \varsigma)}(x)$  on the non-empty open interval where  ${}^2\Psi'_{(F[\beta]/F, \varsigma)}(x) = 1$ . Indeed, the functions  ${}^2\Psi_{(E/F, \varsigma)}$  and  ${}^2\Psi_{(F[\beta]/F, \varsigma)}$  are related by (7.2.1). It follows that the character  $\phi$  of 5.3 Theorem (4) is  $\theta_0|_{U_{F[\beta]}^l}$ . Let  $\mathbb{A}$  be the canonical ultrametric on  $\mathfrak{E}(F)$ .

**Lemma.** *Let  $\rho \in \mathcal{C}(\mathfrak{a}, \alpha)$  have endo-class  $\mathbf{R}$ . The following are equivalent:*

- (1)  $l_E(\rho) < l$ ;
- (2)  $\mathbb{A}(\mathbf{R}, \Theta_0) < l/p^r$ ;
- (3)  $\rho(y) = \theta_0(y)$ , for all  $y \in U_{F[\beta]}^l$ .

*Proof.* Since  $l \leq m/2$ , the definition of  $\mathbb{A}$  implies that  $\mathbb{A}(\mathbf{R}, \Theta_0) = t/p^r$ , where  $t \geq 0$  is the least integer such that  $\rho$  agrees with  $\theta_0$  on  $U_E^{1+t}$ . All assertions now follow readily.  $\square$

Since  $\mathbb{A}(\Theta, \Theta_0) = l/p^r$ , the characters  $\theta, \theta_0$  do not agree on  $U_{F[\beta]}^l$ . It now follows from 5.3 Theorem (4) that  $\Psi_\Theta = {}^2\Psi_{(F[\beta]/F, \varsigma)}$ , as required for part (2).

Part (3) holds relative to the same choice of  $\beta$ , so we have completed the proof of 7.2 Theorem 1.  $\square$

*Remark.* Applying 6.4 Theorem to the preceding argument, the character  $\phi$  of 5.3 Theorem (4) is trivial if and only if  $l \equiv 0 \pmod{p}$ .

**7.4.** We prove 7.2 Theorem 2. Let  $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$  be a realization of  $\Theta$  for which  $\alpha$  is weakly  $\theta$ -conformal. Set  $E = F[\alpha]$ . Thus  $l = l_E(\theta) = m - w_{E/F}$ . If either  $m > 2w_{E/F}$  or  $w_{E/F} \equiv 0 \pmod{p}$ , we are done (5.3 Theorem (2) or (3)), so assume otherwise. In particular,  $l \not\equiv m \pmod{p}$ . The proposition of 6.3 gives an element  $\beta \in P(\mathfrak{a}, \alpha)$  for which  ${}^2\Psi_{(F[\beta]/F, \varsigma_\Theta)}$  has an odd number of jumps and  $l_{F[\beta]}(\theta) = l$ . So, by 7.2 Theorem 1,  $\Psi_\Theta = {}^2\Psi_{(E/F, \varsigma_\Theta)}$ .  $\square$

**7.5.** We prove 7.2 Theorem 3. By 7.1 Proposition, we may start with a realization  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  of  $\Theta$  such that  $\beta$  is  $\theta$ -conformal. In particular,  $l_{F[\beta]}(\theta) = m - w_{F[\beta]/F}$ . If  $w_{F[\beta]/F} \equiv 0 \pmod{p}$ , the assertion is given by part (1) of 7.2 Theorem 1. If  $w_{F[\beta]/F}$  is not divisible by  $p$  and  $< m/2$ , there is again nothing to do. Otherwise, 6.3 Proposition gives a pair  $(E/F, \varsigma_\Theta)$  for which  $l_E(\theta) = l_{F[\beta]}(\theta) \not\equiv m \pmod{p}$ .  $\square$

## 8. Representations with a single jump

We consider here representations  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  for which the decomposition function  $\Sigma_\sigma$  of (2.2.2) has a *unique jump*: these play a central role in what follows.

**8.1.** For the moment, let  $G$  be a finite  $p$ -group with centre  $Z \neq G$ . Say that  $G$  is *H-cyclic* if  $Z$  is cyclic and  $G/Z$  is elementary abelian. Equivalently,  $G$  is an extra-special  $p$ -group of class 2. We introduce this new terminology to avoid ambiguous usage that has accumulated here.

If  $G$  is H-cyclic, the commutator group  $[G, G]$  is the subgroup  $Z_p$  of  $Z$  of order  $p$ . We may view the pairing  $G/Z \times G/Z \rightarrow Z_p$ , induced by the commutator  $(x, y) \mapsto [x, y]$ , as an alternating form on the  $\mathbb{F}_p$ -vector space  $G/Z$ . If  $x, y \in G$ , then  $[x, y] = 1$  if and only if  $x$  centralizes  $y$ . It follows that this pairing is *nondegenerate*: if  $[x, y] = 1$  for all  $y \in G$ , then  $x \in Z$ . The familiar properties of Heisenberg representations apply here.

**Proposition.** *Let  $G$  be an H-cyclic finite  $p$ -group with centre  $Z$ , and let  $\chi$  be a faithful character of  $Z$ .*

- (1) *There exists a unique irreducible representation  $\sigma$  of  $G$  such that  $\sigma|_Z$*

contains  $\chi$ . The representation  $\sigma$  is faithful of dimension  $(G:Z)^{\frac{1}{2}}$  and  $\sigma|_Z$  is a multiple of  $\chi$ .

- (2) A character  $\xi$  of  $G$  satisfies  $\xi \otimes \sigma \cong \sigma$  if and only if  $\xi$  is trivial on  $Z$ . If  $D(\sigma)$  denotes the group of such characters, then

$$(8.1.1) \quad \check{\sigma} \otimes \sigma = \sum_{\xi \in D(\sigma)} \xi.$$

*Proof.* This is a standard exercise.  $\square$

While in this general context, we record a technical result for use in 8.3.

**Lemma.** *Let  $G$  be an  $H$ -cyclic finite  $p$ -group with centre  $Z$ . Let  $\alpha$  be an automorphism of  $G$  which is trivial on  $Z$  and induces the trivial automorphism of  $G/Z$ . The automorphism  $\alpha$  is then inner.*

*Proof.* Let  $Z_p = [G, G]$  be the subgroup of  $Z$  of order  $p$ . Consider the map  $G \rightarrow Z_p$  given by  $x \mapsto x^\alpha x^{-1}$ . This induces a map  $G/Z \rightarrow Z_p$  which is a homomorphism:  $(xy)^\alpha y^{-1} x^{-1} = x^\alpha x^{-1} y^\alpha y^{-1}$ . The nondegeneracy property of the commutator pairing gives a unique  $y \in G/Z$  such that  $x^\alpha x^{-1} = [y, x]$ , for all  $x$ . This relation says  $x^\alpha = yxy^{-1}$ , as required.  $\square$

**8.2.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ , and let  $\bar{\sigma} : \mathcal{W}_F \rightarrow \text{PGL}_{p^r}(\mathbb{C})$  be the projective representation defined by  $\sigma$ . The *centric field*  $Z = Z_\sigma/F$  of  $\sigma$  is defined by  $\mathcal{W}_Z = \text{Ker } \bar{\sigma}$ . The *tame centric field*  $T_\sigma/F$  of  $\sigma$  is the maximal tame sub-extension of  $Z_\sigma/F$ . Thus  $\sigma$  is absolutely wild if and only if  $T_\sigma = F$ . Observe that, if  $K/F$  is a finite tame extension and  $\sigma_K = \sigma|_{\mathcal{W}_K} \in \widehat{\mathcal{W}}_K^{\text{wr}}$ , then  $Z_{\sigma_K} = Z_\sigma K$  and  $T_{\sigma_K} = T_\sigma K$ .

Define  $D(\sigma)$  to be the group of characters  $\chi$  of  $\mathcal{W}_F$  such that  $\chi \otimes \sigma \cong \sigma$ .

Since  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ , the restriction  $\sigma_0^+ = \sigma|_{\mathcal{P}_F}$  is irreducible. Let  $D_0(\sigma)$  be the group of characters  $\phi$  of  $\mathcal{P}_F$  such that  $\phi \otimes \sigma_0^+ \cong \sigma_0^+$ . Since  $\sigma_0^+$  factors through a representation of a finite  $p$ -group, the group  $D_0(\sigma)$  is non-trivial. A character  $\phi$  of  $\mathcal{P}_F$  lies in  $D_0(\sigma)$  if and only if it is a component of  $\check{\sigma}_0^+ \otimes \sigma_0^+$ , whence  $D_0(\sigma)$  has order at most  $p^{2r}$ . The group  $\mathcal{W}_F$  acts on  $D_0(\sigma)$  in a natural way, with  $\mathcal{P}_F$  acting trivially.

If  $K/F$  is a finite tame extension, then  $\mathcal{P}_K = \mathcal{P}_F$ . We may identify  $(\sigma_K)_0^+$  with  $\sigma_0^+$  and  $D_0(\sigma_K)$  with  $D_0(\sigma)$ .

**Lemma.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ .*

- (1) *If  $K/F$  is a finite, tamely ramified field extension, the restriction map  $D(\sigma_K) \rightarrow D_0(\sigma)$  is an isomorphism of  $D(\sigma_K)$  with the group of  $\mathcal{W}_K$ -fixed points in  $D_0(\sigma)$ .*
- (2) *There is a unique minimal tame extension  $T_I(\sigma)/F$  such that the map  $D(\sigma_{T_I(\sigma)}) \rightarrow D_0(\sigma)$  is an isomorphism.*
- (3) *The field extension  $T_I(\sigma)/F$  is Galois and contained in  $T_\sigma$ .*

*Proof.* The lemma summarizes the discussion in [13] 8.2.  $\square$

We sometimes refer to  $T_I(\sigma)$  as the *imprimitivity field* of  $\sigma$ .

**Proposition.** *If  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  is  $H$ -cyclic then  $T_I(\sigma) = T_\sigma$ .*

*Proof.* Let  $Z_\sigma/F$  be the centric field of  $\sigma$ . Since  $T_\sigma$  contains  $T_I(\sigma)$ , nothing is changed if we extend the base field to  $T_I(\sigma)$  and assume  $T_I(\sigma) = F$ . According to the lemma, the group  $D(\sigma)$  is then isomorphic to  $D_0(\sigma)$  and so has order  $p^{2r}$ , where  $p^r = \dim \sigma$ . The non-trivial characters in  $D(\sigma)$  are wildly ramified of order  $p$ . The sum  $\sum_{\phi \in D(\sigma)} \phi$  is a sub-representation of  $\sigma \otimes \check{\sigma}$ , of the same dimension, so  $\check{\sigma} \otimes \sigma = \sum_{\phi \in D(\sigma)} \phi$ . However,  $\check{\sigma} \otimes \sigma$  provides a faithful representation of  $\text{Gal}(Z_\sigma/F)$ : an element of the kernel of  $\check{\sigma} \otimes \sigma$  is necessarily scalar and so lies in  $\text{Ker } \bar{\sigma}$ .

Define  $K/F$  by  $\mathcal{W}_K = \bigcap_{\phi \in D(\sigma)} \text{Ker } \phi$ . The extension  $K/F$  is totally wildly ramified, and elementary abelian of degree  $p^{2r}$ . By definition, every  $\phi \in D(\sigma)$  is trivial on  $\text{Gal}(Z_\sigma/K)$ , whence  $K = Z_\sigma$  and  $T_\sigma = F$ .  $\square$

*Remark.* Following the proposition, it is natural to ask whether there exists a representation  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  for which the tame centric field and the imprimitivity field are distinct. We show how to produce such representations  $\sigma$  in 9.7 below.

The following device is not central to our current concerns, but we include it here for its utility in constructing examples (as in 8.4 below).

**Example.** *Let  $\sigma, \sigma' \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be  $H$ -cyclic. The following are equivalent:*

- (1)  $D_0(\sigma) \cap D_0(\sigma') = \{1\}$ ;
- (2)  $\sigma \otimes \sigma'$  is irreducible and totally wild.

*When these conditions hold, the representation  $\sigma \otimes \sigma'$  is  $H$ -cyclic.*

*Proof.* If  $\tau$  is a smooth, finite-dimensional representation of  $\mathcal{P}_F$ , then  $\tau$  is irreducible if and only if the space  $\text{Hom}_{\mathcal{P}_F}(1, \tau \otimes \check{\tau})$  has dimension one. Here,



$\sigma \otimes \check{\sigma} \mid \mathcal{P}_F = \sum_{\phi \in D_0(\sigma)} \phi$ , and similarly for  $\sigma'$ . Therefore

$$(\sigma \otimes \sigma') \otimes (\check{\sigma} \otimes \check{\sigma}') \mid \mathcal{P}_F = \sum_{\substack{\phi \in D_0(\sigma), \\ \phi' \in D_0(\sigma')}} \phi \phi'.$$

The trivial character occurs exactly once in the sum if and only if  $D_0(\sigma) \cap D_0(\sigma') = \{1\}$ , so (1) is equivalent to  $\sigma \otimes \sigma'$  being irreducible on  $\mathcal{P}_F$ : this is the same as (2).

Abbreviate  $\tau = \sigma \otimes \sigma'$ , and assume  $\tau$  to be irreducible. For  $x \in \mathcal{P}_F$ , the operator  $\tau(x)^p$  is scalar and lies in  $CC'$ , where  $C, C'$  are the centres of  $\sigma(\mathcal{P}_F), \sigma'(\mathcal{P}_F)$  respectively. In particular,  $CC'$  consists of scalars and is central in  $\tau(\mathcal{P}_F)$ . Thus  $\tau(\mathcal{P}_F)$  is of exponent  $p$  modulo its centre. Since  $\tau$  is irreducible on  $\mathcal{P}_F$ , this centre is cyclic.  $\square$

**8.3.** Let  $\chi$  be a character of  $\mathcal{P}_F$ . Define the  $F$ -slope  $\text{sl}_F(\chi)$  of  $\chi$  by

$$(8.3.1) \quad \text{sl}_F(\chi) = \inf \{x > 0 : \mathcal{R}_F(x) \subset \text{Ker } \chi\}.$$

If  $\chi$  extends to a character  $\tilde{\chi}$  of  $\mathcal{W}_F$ , then  $\text{sl}_F(\chi) = \text{sw}(\tilde{\chi}) = \varsigma_{\tilde{\chi}}$ .

Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be  $H$ -cyclic, with  $\dim \sigma > 1$ . Say that  $\sigma$  is  $H$ -singular if there exists  $a > 0$  such that  $\text{sl}_F(\chi) = a$ , for all non-trivial  $\chi \in D_0(\sigma)$ .

**Proposition 1.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be  $H$ -singular and let  $a = \text{sl}_F(\chi)$ , for  $\chi \in D_0(\sigma)$ ,  $\chi \neq 1$ . The function  $\Sigma_\sigma$  has a unique jump, lying at the point  $a$ .*

*Proof.* This is immediate, on applying (2.2.2) to  $\sigma$  and (8.1.1) to  $\sigma \mid \mathcal{P}_F$ .  $\square$

The converse is more interesting.

**Proposition 2.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  have dimension  $p^r$ ,  $r \geq 1$ . Suppose that the decomposition function  $\Sigma_\sigma(x)$  has exactly one jump, at the point  $a$ , say. The following properties then hold:*

- (1) *the representation  $\sigma$  is  $H$ -singular and  $\text{sl}_F(\chi) = a$ , for every  $\chi \in D_0(\sigma)$ ,  $\chi \neq 1$ ;*
- (2)  *$\text{sw}(\check{\sigma} \otimes \sigma) = p^{2r} \Sigma_\sigma(0) = (p^{2r} - 1)a$ ;*
- (3) *if  $\sigma$  is of Carayol type, then  $a = \text{sw}(\sigma)/(1+p^r)$ .*

*Proof.* Applying a suitable tamely ramified base field extension, we reduce to the case where  $\sigma$  is absolutely wild (cf. 3.2 Definition and Lemma). In particular,  $D(\sigma) \cong D_0(\sigma)$  (8.2 Lemma) and  $\text{sl}_F(\chi \mid \mathcal{P}_F) = \text{sw}(\chi)$ , for all  $\chi \in D(\sigma)$ .

The definition (2.2.2) of  $\Sigma_\sigma$  implies first that

$$(8.3.2) \quad \Sigma'_\sigma(x) = \begin{cases} p^{-2r}, & 0 < x < a, \\ 1, & x > a, \end{cases}$$

and second that the restriction of  $\sigma$  to  $\mathcal{R}_F(a)$  is irreducible, while its restriction to  $\mathcal{R}_F^+(a)$  is a multiple of a character. The group  $S^+ = \sigma(\mathcal{R}_F^+(a))$  is therefore cyclic and central in  $S = \sigma(\mathcal{R}_F(a))$ . The finite  $p$ -group  $S/S^+$  is a quotient of  $\mathcal{R}_F(a)/\mathcal{R}_F^+(a)$ , so it is elementary. Since  $\sigma$  is irreducible on  $\mathcal{R}_F(a)$ , the centre of  $S$  is cyclic. Consequently, the group  $S = \sigma(\mathcal{R}_F(a))$  is H-cyclic with centre containing  $S^+$ .

Let  $y \in \mathcal{P}_F$ . The representations  $\sigma, \sigma^y$  are equivalent, particularly on  $\mathcal{R}_F(a)$ . The element  $y$  must therefore act trivially on the centre of  $S$ . The commutator group  $[y, \mathcal{R}_F(a)]$  is contained in  $[\mathcal{P}_F, \mathcal{R}_F(a)] \subset \mathcal{R}_F^+(a)$ , so  $y$  acts trivially on  $S$  modulo its centre. By 8.1 Lemma, there exists  $x \in \mathcal{R}_F(a)$  such that  $\sigma(xy)$  centralizes  $S$ . Therefore  $\sigma(\mathcal{P}_F)$  is the product of  $S$  and the  $\sigma(\mathcal{P}_F)$ -centralizer of  $S$ . Since  $\sigma$  is irreducible on  $\mathcal{R}_F(a)$ , this centralizer is cyclic, whence  $\sigma(\mathcal{P}_F)$  is H-cyclic.

Recall from (8.1.1) that  $\check{\sigma} \otimes \sigma = \sum_{\chi \in D(\sigma)} \chi$ . A non-trivial character  $\chi \in D(\sigma)$  is non-trivial on  $\mathcal{P}_F$  but it is trivial on the centre of  $\sigma(\mathcal{P}_F)$ . It is therefore determined by its restriction to  $\mathcal{R}_F(a)$ . It is certainly trivial on  $\mathcal{R}_F^+(a)$  so it has Swan exponent  $a$ . Thus  $\sigma$  is H-singular and (1) is proven. Part (2) now follows from (8.1.1). Part (3) is 3.8 Proposition.  $\square$

**8.4.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be H-singular, and let  $a$  be the jump of  $\Sigma_\sigma$ . In the proof of 8.3 Proposition 2, we saw that  $\sigma(\mathcal{P}_F)$  is the product of  $\sigma(\mathcal{R}_F(a))$  and its centralizer in  $\sigma(\mathcal{P}_F)$ . The key point of the argument is that  $\mathcal{P}_F$  acts trivially on  $\sigma(\mathcal{R}_F(a))$  modulo its centre, so that 8.1 Lemma applies. This group, recall, is dual to  $D_0(\sigma)$ . Combining this with the definition of the imprimitivity field  $T_I(\sigma)$  and 8.2 Proposition, we get:

**Corollary.** *Let  $Z = Z_\sigma$ ,  $T = T_\sigma = T_I(\sigma)$ .*

(1) *The field  $Z$  is given by*

$$\mathcal{W}_Z = \bigcap_{\chi \in D(\sigma_T)} \text{Ker } \chi.$$

(2) *The Herbrand function  $\psi_{Z/T}$  has a unique jump, lying at  $e(T|F)a$ . Moreover,*

- (a)  $\mathcal{R}_F^+(a) \subset \mathcal{W}_Z$  and
- (b)  $\mathcal{W}_T = \mathcal{R}_F(a)\mathcal{W}_Z$ .
- (3) The group  $\sigma(\mathcal{W}_T)$  is the centralizer in  $\sigma(\mathcal{W}_F)$  of  $\sigma(\mathcal{R}_F(a))$ .

We finish with an example derived from [11] and 8.2 Example.

**Example.** Take  $p = 2$ , and suppose that  $F$  contains a primitive cube root of unity. For  $i = 1, 2$ , let  $\sigma_i \in \widehat{\mathcal{W}}_F^{\text{wr}}$  have dimension 2 and satisfy  $\text{sw}(\sigma_i) = 1$ . Theorem 5.1 of [11] gives the recipe for  $T_I(\sigma_i)$  and  $D_0(\sigma_i)$ . From that information and 8.2 Example, one sees it is possible to choose  $\sigma_1, \sigma_2$  so that  $\sigma = \sigma_1 \otimes \sigma_2$  is irreducible and H-singular. It is not of Carayol type, as  $\text{sw}(\sigma) = 2$ . If  $[\sigma]_0^+ = {}^L\Theta$ ,  $\Theta \in \mathcal{E}(F)$ , then  $\Psi_\Theta$  has two jumps and is not convex: see 8.5 Example 1 of [13] for the formula.

## 9. Ramification structure

Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type. We return to the methods of section 3 to work out the structure of  $\sigma$  when restricted to an arbitrary ramification group of  $\mathcal{W}_F$ . If  $[\sigma]_0^+ = {}^L\Theta$ ,  $\Theta \in \mathcal{E}^C(F)$ , we get a formula for  $\Psi_\Theta$  to set against those of section 7. Appearances to the contrary, everything in this section relies on the local Langlands correspondence and the conductor formula of [14], since we use the main results of section 3.

**9.1.** To avoid carrying an irrelevant variable, we make a minor adjustment to our notation. If  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  and if  $\Theta \in \mathcal{E}(F)$  satisfies  $[\sigma]_0^+ = {}^L\Theta$ , we now write  $\Psi_\sigma = \Psi_\Theta$  and use whichever form is convenient at the time.

Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type, and set  $\varsigma = \varsigma_\sigma$ . If  $0 < x < \varsigma$ , define

$$(9.1.1) \quad w_\sigma(x) = \lim_{\epsilon \rightarrow 0} \Psi'_\sigma(x+\epsilon)/\Psi'_\sigma(x-\epsilon).$$

Thus  $w_\sigma(x)$  is a non-negative power of  $p$ , and  $w_\sigma(x) > 1$  if and only if  $x$  is a jump of  $\Psi_\sigma$ . We then sometimes call  $w_\sigma(x)$  the *height* of the jump  $x$ . (This notation is inconsistent with the usage of section 3, for typographical reasons.)

The functional equation (3.1.1) gives an order-reversing involution  $j \mapsto \bar{j}$  on the set of jumps of  $\Psi_\sigma$ . If  $\Psi_\sigma$  has an even number of jumps, this involution has no fixed point. If the number of jumps is odd, it fixes the central one. In the notation of (9.1.1), the symmetry property of  $\Psi_\sigma$  gives

$$(9.1.2) \quad w_\sigma(\bar{j}) = w_\sigma(j).$$

We will occasionally have to deal with the case of a one-dimensional representation  $\sigma$ . There,  $\Sigma_\sigma(x) = \Psi_\sigma(x) = x$  and the functions  $\Sigma_\sigma, \Psi_\sigma$  have no jumps. Indeed, the converse also holds [13] 7.7.

**9.2.** We state the main results of the section. We use the following notation.

**Notation.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type and dimension  $p^r$ ,  $r \geq 1$ . Define  $c_\sigma$  by  $c_\sigma + \Psi_\sigma(c_\sigma) = \varsigma_\sigma$ . Let

$$(9.2.1) \quad j_1 < j_2 < \cdots < j_s < (c_\sigma) < \bar{j}_s < \bar{j}_{s-1} < \cdots < \bar{j}_1$$

be the jumps of  $\Psi_\sigma$  with the understanding that

- (a) the term  $c_\sigma$  is included only if  $\Psi_\sigma$  has an odd number of jumps and
- (b)  $s = 0$  when  $\Psi_\sigma$  has only one jump.

Initially, we treat the *absolutely wild* case  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  (3.2 Definition). In 9.5, we extend the results to the totally wild case  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ .

**Theorem.** Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  be of Carayol type and dimension  $p^r$ .

- (1) The restriction  $\sigma|_{\mathcal{R}_F^+(c_\sigma)}$  is a direct sum of characters.
- (2) Let  $\xi$  be a character of  $\mathcal{R}_F^+(c_\sigma)$  occurring in  $\sigma$  and let  $\mathcal{W}_{L_\xi}$  be the  $\mathcal{W}_F$ -stabilizer of  $\xi$ . Let  $\sigma_\xi$  be the natural representation of  $\mathcal{W}_{L_\xi}$  on the  $\xi$ -isotypic subspace of  $\sigma$ .
  - (a) The field extension  $L_\xi/F$  is absolutely wildly ramified (cf. 1.2) of degree  $p^r w_\sigma(c_\sigma)^{-\frac{1}{2}}$  and  $\mathcal{W}_{L_\xi}$  contains  $\mathcal{R}_F^+(c_\sigma)$ .
  - (b) The representation  $\sigma_\xi$  is irreducible, absolutely wild and

$$\sigma = \text{Ind}_{L_\xi/F} \sigma_\xi.$$

- (c) If  $c_\sigma$  is not a jump of  $\Psi_\sigma$ , then  $\sigma_\xi$  is a character. Otherwise,  $\sigma_\xi$  is  $H$ -singular, of Carayol type and dimension  $w_\sigma(c_\sigma)^{\frac{1}{2}}$ . The unique jump of  $\Psi_{\sigma_\xi}$  lies at  $\psi_{L_\xi/F}(c_\sigma)$ .

*Remarks.*

- (1) The triple  $(\xi, L_\xi, \sigma_\xi)$  is uniquely determined by  $\sigma$ , up to  $\mathcal{W}_F$ -conjugation.
- (2) The function  $\Psi_\sigma$  has no jump lying strictly between  $c_\sigma$  and  $\bar{j}_s$ . So, if  $\xi, \xi'$  are components of  $\sigma|_{\mathcal{R}_F^+(c_\sigma)}$ , then  $\xi = \xi'$  if and only if  $\xi|_{\mathcal{R}_F(\bar{j}_s)} = \xi'|_{\mathcal{R}_F(\bar{j}_s)}$ . If  $c_\sigma$  is not a jump then, in the same way,  $\sigma|_{\mathcal{R}_F^+(j_s)}$  is a sum of characters, two of which are equal if and only if their restrictions to  $\mathcal{R}_F(\bar{j}_s)$  are equal.

As we prove the theorem, we uncover further features of interest which we now list.

**Complement 1.** *Let  $1 \leq k \leq s$ .*

- (1) *The restriction  $\sigma \mid \mathcal{R}_F^+(j_k)$  is a multiplicity-free direct sum of irreducible representations.*
- (2) *The restriction  $\sigma \mid \mathcal{R}_F(\bar{j}_k)$  is a direct sum of characters. The isotypic components of  $\sigma \mid \mathcal{R}_F(\bar{j}_k)$  are the subspaces  $\tau \mid \mathcal{R}_F(\bar{j}_k)$ , as  $\tau$  ranges over the irreducible components of  $\sigma \mid \mathcal{R}_F^+(j_k)$ .*

In light of Remark (2) above, one can equally relate the decompositions of  $\sigma \mid \mathcal{R}_F(j_k)$  and  $\sigma \mid \mathcal{R}_F^+(\bar{j}_k)$ . In the next result, we use the concept of *elementary resolution* from 1.9.

**Complement 2.** *For  $1 \leq k \leq s$ , choose an irreducible component  $\tau_k$  of the restriction  $\sigma \mid \mathcal{R}_F^+(j_k)$  so that  $\tau_{k+1}$  is a component of  $\tau_k \mid \mathcal{R}_F^+(j_{k+1})$ ,  $1 \leq k < s$ . Let  $\mathcal{W}_{E_k}$  be the  $\mathcal{W}_F$ -stabilizer of  $\tau_k$ . If  $\xi$  is a character of  $\mathcal{R}_F^+(c_\sigma)$  occurring in  $\tau_s \mid \mathcal{R}_F^+(c_\sigma)$ , then  $E_s = L_\xi$ . The tower of fields*

$$(9.2.2) \quad F \subset E_1 \subset E_2 \subset \dots \subset E_s = L_\xi$$

*is the elementary resolution of  $L_\xi/F$  and*

$$(9.2.3) \quad \Psi_\sigma(x) = p^{-r} \psi_{L_\xi/F}(x), \quad 0 \leq x \leq c_\sigma.$$

Because of symmetry (4.1 Proposition (1)), the relation (9.2.3) determines  $\Psi_\sigma(x)$  for  $0 \leq x \leq c_\sigma$ . The tower of fields (9.2.2) is uniquely determined by  $\sigma$ , up to  $\mathcal{W}_F$ -conjugacy.

**9.3.** The theorem of 9.2 is proved by induction on the number of jumps. If  $\Psi_\sigma$  has no jumps, then  $\dim \sigma = 1$  and this case has been excluded. If  $\Psi_\sigma$  has just one jump, the results follow from the discussion and proposition of 8.3 with  $L_\xi = F$ . In this sub-section, we assume that  $\Psi_\sigma$  has at least two jumps and give a general reduction step.

**Proposition.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{awr}}$  be of Carayol type. Suppose that  $\Psi_\sigma$  has at least two jumps, of which  $a$  is the first and  $z = \bar{a}$  the last.*

*Let  $T_a(\sigma)$  be the group of  $\chi \in D(\sigma)$  for which  $\text{sw}(\chi) \leq a$ . Let  $E_1/F$  be class field to  $T_a(\sigma)$ .*

- (1) *The group  $T_a(\sigma)$  is elementary abelian of order  $w_\sigma(a)$ .*

- (2) The group  $\mathcal{R}_F^+(a)$  is contained in  $\mathcal{W}_{E_1}$  and  $\mathcal{W}_F = \mathcal{R}_F(a)\mathcal{W}_{E_1}$ .
- (3) There exists  $\sigma_1 \in \widehat{\mathcal{W}}_{E_1}^{\text{awr}}$  such that  $\sigma = \text{Ind}_{E_1/F} \sigma_1$ . Moreover,

$$(9.3.1) \quad \sigma|_{\mathcal{R}_F^+(a)} = \sum_{\gamma \in \text{Gal}(E_1/F)} \sigma_1^\gamma|_{\mathcal{R}_F^+(a)}.$$

The representations  $\sigma_1^\gamma|_{\mathcal{R}_F^+(a)}$ ,  $\gamma \in \text{Gal}(E_1/F)$ , are distinct and irreducible. The  $\mathcal{W}_F$ -stabilizer of  $\sigma_1|_{\mathcal{R}_F^+(a)}$  is  $\mathcal{W}_{E_1}$ .

- (4) The jumps of  $\Psi_{\sigma_1}$  are  $\psi_{E_1/F}(j)$ , as  $j$  ranges over the jumps of  $\Psi_\sigma$ ,  $j \neq a, z$ . Indeed,  $w_\sigma(y) = w_{\sigma_1}(\psi_{E_1/F}(y))$ , for  $y \neq a, z$ .
- (5) The restriction  $\sigma|_{\mathcal{R}_F(z)}$  is a direct sum of characters  $\xi$ . The  $\mathcal{W}_F$ -stabilizer of any such  $\xi$  is  $\mathcal{W}_{E_1}$ .

*Proof.* The group  $T_a(\sigma)$  is non-trivial (3.3 Lemma 1), so choose  $\chi \in T_a(\sigma)$ ,  $\chi \neq 1$ . Set  $\mathcal{W}_K = \text{Ker } \chi$ . The extension  $K/F$  is cyclic of degree  $p$ , and  $\psi_{K/F}$  has a unique jump, lying at  $a$ . As in 3.3 Lemma 2,  $\mathcal{W}_K \cap \mathcal{R}_F(a) = \mathcal{R}_K(a)$  is of index  $p$  in  $\mathcal{R}_F(a)$  and  $\mathcal{R}_K^+(a) = \mathcal{R}_F^+(a)$ . There exists  $\tau \in \widehat{\mathcal{W}}_K^{\text{wr}}$  with  $\sigma = \text{Ind}_{K/F} \tau$ , the representation  $\tau$  being absolutely wild of Carayol type (3.3 Lemma 1) or a character. By 3.4 Theorem and 3.5 Lemma 4,  $w_\sigma(y) = w_\tau(\psi_{K/F}(y))$ , provided  $y \neq a, z$ . On the other hand,  $w_\sigma(a) = pw_\tau(a)$  and  $w_\sigma(z) = pw_\tau(\psi_{K/F}(z))$  (3.4 Theorem and (3.5.3)): because of 3.5 Corollary 1, the hypothesis on the jumps of  $\sigma$  excludes possibility (3) of 3.4 Theorem.

**Lemma 1.**

- (1) The restriction  $\tau|_{\mathcal{R}_K(a)}$  is irreducible and

$$(9.3.2) \quad \sigma|_{\mathcal{R}_F^+(a)} = \sum_{\delta \in \text{Gal}(K/F)} \tau^\delta|_{\mathcal{R}_F^+(a)}.$$

- (2) The representations  $\tau^\delta|_{\mathcal{R}_F^+(a)}$ ,  $\delta \in \text{Gal}(K/F)$ , are disjoint.

*Proof.* Since  $a$  is the first jump of  $\Sigma_\sigma$ , the restriction  $\sigma|_{\mathcal{R}_F(a)}$  is irreducible. The Mackey formula implies that this restriction is  $\text{Ind}_{\mathcal{R}_K(a)}^{\mathcal{R}_F(a)} \tau|_{\mathcal{R}_K(a)}$ , whence the first assertion follows. The relation (9.3.2) again follows from the Mackey formula. In (2), the restrictions of the  $\tau^\delta$  are either disjoint or identical, and (3.5.1) implies they are disjoint.  $\square$

We now proceed by induction on the integer  $w_\sigma(a)$ , starting with the case  $w_\sigma(a) = p$ . Thus  $w_\tau(a) = 1$ . We prove the proposition for  $E_1 = K$  and  $\sigma_1 = \tau$ .

The hypothesis implies that  $a$  is not a jump of  $\Sigma_\tau$  (giving point (4)), and so  $\tau$  is irreducible on  $\mathcal{R}_F^+(a) = \mathcal{R}_K^+(a)$ . It follows that  $D(\tau)$  has no element of Swan exponent  $a$  so  $T_a(\sigma)$  has order  $p = w_\sigma(a)$ . The point  $\psi_{K/F}(z)$  is not a jump of  $\Sigma_\tau$ , by symmetry applied to  $\Psi_\tau$ . It follows that  $\tau|_{\mathcal{R}_F(z)}$  is a multiple of a character. Thus

$$\sigma|_{\mathcal{R}_F(z)} = \sum_{\delta \in \text{Gal}(K/F)} \tau^\delta|_{\mathcal{R}_F(z)}$$

is a sum of characters. Since  $z$  is a jump of  $\Psi_\sigma$ , these characters cannot all be the same: they fall into  $p$  distinct orbits under  $\mathcal{W}_F$ . Assertion (5) follows, and the proof is complete in the case  $w_\sigma(a) = p$ .

Suppose next that  $w_\sigma(a)$  is divisible by  $p^2$ . In particular,  $\tau$  is not a character. Inductively, we may assume that the result holds for the representation  $\tau \in \widehat{\mathcal{W}}_K^{\text{awr}}$ . That is:

**Lemma 2 (inductive hypothesis).** *Let  $E/K$  be class field to the group  $T_a(\tau)$ . Let  $\zeta \in \widehat{\mathcal{W}}_E^{\text{awr}}$  satisfy  $\text{Ind}_{E/K} \zeta = \tau$ .*

- (1) *The group  $T_a(\tau)$  is elementary abelian of order  $w_\tau(a)$ .*
- (2) *The group  $\mathcal{R}_K^+(a)$  is contained in  $\mathcal{W}_E$  and  $\mathcal{W}_K = \mathcal{R}_K(a)\mathcal{W}_E$ .*
- (3) *In the expansion*

$$(9.3.3) \quad \tau|_{\mathcal{R}_K^+(a)} = \sum_{\gamma \in \text{Gal}(E/K)} \zeta^\gamma|_{\mathcal{R}_K^+(a)},$$

*the terms  $\zeta^\gamma|_{\mathcal{R}_K^+(a)}$ ,  $\gamma \in \text{Gal}(E/K)$ , are distinct and irreducible. The  $\mathcal{W}_K$ -stabilizer of  $\zeta|_{\mathcal{R}_K^+(z)}$  is  $\mathcal{W}_E$ .*

- (4) *We have  $w_\zeta(a) = w_\zeta(\psi_{E/F}(z)) = 1$  while  $w_\zeta(\psi_{E/K}(y)) = w_\tau(y)$ ,  $y \neq a, \psi_{K/F}(z)$ .*
- (5) *The restriction of  $\tau$  to  $\mathcal{R}_F(z) = \mathcal{R}_K(\psi_{K/F}(z))$  is a direct sum of characters  $\xi$ . The  $\mathcal{W}_K$ -stabilizer of any such  $\xi$  is  $\mathcal{W}_E$ .*

Taking  $E/K$  as in Lemma 2, we prove that  $E/F$  is class field to  $T_a(\sigma)$ . Each of the functions  $\psi_{K/F}$ ,  $\psi_{E/K}$  has a unique jump, lying at  $a$ . The same therefore applies to  $\psi_{E/F}$ . The field  $E$  appears as a subfield of the centric field of  $\sigma$ , so  $E/F$  is absolutely wild and we may apply 1.9 Proposition. This implies that  $E/F$  is elementary abelian and every element  $\phi$  of  $T_a(E|K)$  extends to a character  $\tilde{\phi}$  of  $\mathcal{W}_F$  lying in  $T_a(E|F)$ . We have  $\tilde{\phi} \otimes \sigma = \text{Ind}_{K/F} \phi \otimes \tau = \text{Ind}_{K/F} \tau = \sigma$ . That is,  $\tilde{\phi} \in T_a(\sigma)$ , whence  $T_a(\sigma) = T_a(E|F)$  and this group has order  $pw_\tau(a) = w_\sigma(a)$ .

We have proved part (1) of the proposition, with  $E_1 = E$ . Part (2) follows from the relation  $\psi_{E/F}(a) = a$ . We have already noted in Lemma 1 that the representations

$$\left( \sum_{\gamma \in \text{Gal}(E/K)} \zeta^\gamma \mid \mathcal{R}_F^+(a) \right)^\delta, \quad \delta \in \text{Gal}(K/F),$$

are disjoint. Part (3) of the proposition now follows by induction.

Part (4) of the proposition follows from part (4) of the lemma, by induction. It remains to prove part (5). By part (5) of the lemma,  $\sigma \mid \mathcal{R}_F(z)$  is a sum of characters. The representations  $\tau^\delta$ ,  $\delta \in \text{Gal}(K/F)$ , are disjoint on  $\mathcal{R}_F(z)$  by (3.5.1). The result follows by induction, with  $E_1 = E$  and  $\sigma_1 = \zeta$ .  $\square$

**9.4.** We prove 9.2 Theorem and its complements. We proceed by induction on the number of jumps of  $\Psi_\sigma$ .

*Proof of Theorem.* When  $\Psi_\sigma$  has at most one jump, there is nothing more to say. We therefore assume, in the notation (9.2.1), that  $s \geq 1$ . We apply 9.3 Proposition to get a Galois extension  $E_1/F$  and a representation  $\sigma_1 \in \widehat{\mathcal{W}}_{E_1}^{\text{awr}}$  such that  $\sigma = \text{Ind}_{E_1/F} \sigma_1$ . The extension  $E_1/F$  has a unique jump, lying at  $j_1$ , so  $\mathcal{R}_F^+(x) \subset \mathcal{W}_{E_1}$  for  $x \geq j_1$ . The function  $\Psi_{\sigma_1}$  has jumps at  $\psi_{E_1/F}(j)$ , where  $j$  ranges over all jumps of  $\Psi_\sigma$ , subject to  $j \neq j_1, \bar{j}_1$ .

Suppose the number of jumps to be even. By induction,  $\sigma_1 \mid \mathcal{R}_F^+(j_s)$  is a sum of characters, so the same applies to  $\sigma \mid \mathcal{R}_F^+(j_s)$ . Part (1) is done in this case. The field  $L = L_\xi$  appears as a subfield of the centric field of  $\sigma$ , so  $L/F$  is absolutely wild. The inductive hypothesis gives a character  $\rho_1$  of  $\mathcal{W}_L$  which induces  $\sigma_1$ . It follows that  $\text{Ind}_{L/F} \rho_1 = \sigma$ , and  $\rho_1$  has the necessary properties relative to  $\sigma$ . This proves part (2) of the theorem when the number of jumps is even.

Suppose that the number of jumps is odd. Thus, by induction,  $\sigma_1 \mid \mathcal{R}_{E_1}(c_{\sigma_1})$  is not a sum of characters, while  $\sigma_1 \mid \mathcal{R}_{E_1}^+(c_{\sigma_1})$  is such. Since  $\mathcal{R}_{E_1}(c_{\sigma_1}) = \mathcal{R}_F(\varphi_{E_1/F}(c_{\sigma_1}))$ , the point  $\varphi_{E_1/F}(c_{\sigma_1})$  is a jump of  $\Psi_\sigma$ . That is,  $c_\sigma = \varphi_{E_1/F}(c_{\sigma_1})$  and we have proved part (1) of the theorem. Assertions (2)(a)–(c) now follow by induction, exactly as in the first case, on noting that  $\dim \sigma_\xi = w_\sigma(c_\sigma)^{\frac{1}{2}}$  by 8.2 Proposition.  $\square$

*Proof of Complement 1.* We follow 9.3 Proposition to write  $\sigma = \text{Ind}_{E_1/F} \sigma_1$ . That result also shows that  $\sigma \mid \mathcal{R}_F^+(j_1)$  is multiplicity-free. For  $2 \leq k \leq s$ , the



restriction  $\sigma_1 \mid \mathcal{R}_F^+(j_k)$  is multiplicity-free by inductive hypothesis. The relation  $w_{\sigma_1}(\psi_{E_1/F}(j_k)) = w_\sigma(j_k)$  implied by 3.4 Theorem shows that  $\sigma \mid \mathcal{R}_F^+(j_k)$  is multiplicity-free, and we have proved part (1).

The first assertion of (2) follows from part (1) of the theorem. Since  $\bar{j}_1$  is the last jump of  $\Psi_\sigma$ , the restriction  $\sigma_1 \mid \mathcal{R}_F(\bar{j}_1)$  is a multiple of a character while  $\sigma \mid \mathcal{R}_F(\bar{j}_1)$  is a direct sum of characters. The number of isotypic components in  $\sigma \mid \mathcal{R}_F(\bar{j}_1)$  is  $w_\sigma(\bar{j}_1) = w_\sigma(j_1) = [E_1:F]$ , by 9.3 Proposition, whence the result follows.  $\square$

*Proof of Complement 2.* Recall that  $E_1/F$  was defined in 9.3 as class field to the group  $T_{j_1}(\sigma)$  of characters  $\chi$  of  $\mathcal{W}_F$  such that  $\chi \otimes \sigma \cong \sigma$  and  $\text{sw}(\chi) \leq j_1$ . Thus  $E_1/F$  is Galois and, by 9.3 Proposition (3),  $\mathcal{W}_{E_1}$  is the  $\mathcal{W}_F$ -stabilizer of any irreducible component of  $\sigma \mid \mathcal{R}_F^+(j_1)$ . In the first instance, we may therefore choose the extension  $L = L_\xi/F$  of the theorem, within its conjugacy class, so that  $E_1 \subset L_\xi$ . Since all choices of  $\xi$  are  $\mathcal{W}_F$ -conjugate and  $E_1/L$  is Galois, we have  $E_1 \subset L_\xi$  for all  $\xi$ . That is,  $E_1 \subset L$ .

Because of the relation  $\sigma = \text{Ind}_{L/F} \rho_\xi$ , a character  $\phi$  of  $\mathcal{W}_F$  with  $\phi \mid \mathcal{W}_L$  trivial must satisfy  $\phi \otimes \sigma \cong \sigma$ . The definition of  $E_1$  in 9.3 implies that  $j_1$  is least jump of  $\psi_{L/F}$ . By 1.9 Proposition (2),  $E_1/F$  is the first step in the elementary resolution of  $L/F$ . The first assertion of Complement 2 now follows by induction.

In the proof of the theorem, we showed that  $c_{\sigma_1} = \psi_{E_1/F}(c_\sigma)$ . From 3.4 Theorem we conclude that the jumps of  $\Psi_{\sigma_1}$  are

$$\begin{aligned} \psi_{E_1/F}(j_2) &< \psi_{E_1/F}(j_3) < \cdots < \psi_{E_1/F}(j_s) \\ &< (\psi_{E_1/F}(c_{\sigma_1})) < \psi_{E_1/F}(\bar{j}_s) < \cdots < \psi_{E_1/F}(\bar{j}_2), \end{aligned}$$

with the same convention regarding the central entry in the list. Moreover,

$$(9.4.1) \quad w_{\sigma_1}(\psi_{E_1/F}(j_k)) = w_\sigma(j_k), \quad 2 \leq k \leq s,$$

and similarly relative to the central jump. Let  $w_1 = w_\sigma(j_1)$ , so that  $w_1 = [E_1:F]$ . The functions  $\Psi_\sigma(x)$ ,  $w_1^{-1}\Psi_{\sigma_1}(\psi_{E_1/F}(x))$  have the same jumps in the region  $0 \leq x \leq c_\sigma$ . The heights (9.1) of these jumps are the same, and the functions agree on a region  $0 \leq x < \varepsilon$ . We conclude by induction that

$$\begin{aligned} \Psi_\sigma(x) &= w_1^{-1}\Psi_{\sigma_1}(\psi_{E_1/F}(x)) \\ &= p^{-r}\psi_{L/F}(x), \quad 0 \leq x \leq c_\sigma. \end{aligned}$$

This proves (9.2.3).  $\square$

**9.5.** We extend the results of 9.2 to representations of Carayol type that are totally, but not absolutely, wild. The notational conventions of 9.1, 9.2 remain in force.

**Corollary.** *Let  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  be of Carayol type and dimension  $p^r$ . Define  $c_\sigma$  by the equation  $c_\sigma + \Psi_\sigma(c_\sigma) = \varsigma_\sigma$ .*

(1) *The representation  $\sigma|_{\mathcal{R}_F^+(c_\sigma)}$  is a direct sum of characters.*

*Let  $\xi$  be a character of  $\mathcal{R}_F^+(c_\sigma)$  occurring in  $\sigma$ . Let  $\mathcal{W}_{L_\xi}$  be the  $\mathcal{W}_F$ -stabilizer of  $\xi$  and let  $\sigma_\xi$  be the natural representation of  $\mathcal{W}_{L_\xi}$  on the  $\xi$ -isotypic subspace of  $\sigma|_{\mathcal{R}_F^+(c_\sigma)}$ .*

(2) *The representation  $\sigma_\xi$  is irreducible and  $\text{Ind}_{L_\xi/F} \sigma_\xi = \sigma$ . Moreover,*

(a)  *$\dim \sigma_\xi = w_\sigma(c_\sigma)^{1/2}$ , and*

(b) *if  $\dim \sigma_\xi > 1$ , then  $\sigma_\xi$  is totally wild,  $H$ -singular and of Carayol type.*

(3) *The field extension  $L_\xi/F$  is totally ramified of degree  $p^r / \dim \sigma_\xi$  and*

$$(9.5.1) \quad \Psi_\sigma(x) = p^{-r} \psi_{L_\xi/F}(x), \quad 0 \leq x \leq c_\sigma.$$

*Proof.* Let  $T = T_\sigma/F$  be the tame centric field of  $\sigma$ . Thus  $\tau = \sigma|_{\mathcal{W}_T}$  is absolutely wild of Carayol type. If  $e = e(T|F)$ , then  $\Psi_\sigma(x) = \Psi_\tau(ex)/e$  and  $\varsigma_\tau = e\varsigma_\sigma$ , so  $c_\tau = ec_\sigma$ .

Consequently,  $\mathcal{R}_F^+(c_\sigma) = \mathcal{R}_T^+(c_\tau)$  and part (1) of the corollary follows from part (1) of the theorem. All choices of  $\xi$  are  $\mathcal{W}_F$ -conjugate so let us fix one and write  $L_\xi = L$ . The  $\mathcal{W}_T$ -stabilizer of  $\xi$  is  $\mathcal{W}_T \cap \mathcal{W}_L = \mathcal{W}_{LT}$ . The natural representation of  $\mathcal{W}_{LT}$  on the  $\xi$ -isotypic subspace of  $\tau|_{\mathcal{R}_T^+(c_\tau)}$  is  $\sigma_\xi|_{\mathcal{W}_{LT}}$ , which is irreducible. It follows that  $\sigma_\xi$  is irreducible and has properties (2)(a), (2)(b). Moreover,  $\mathcal{R}_F(c_\sigma)$  is contained in  $\mathcal{W}_L$  and  $\sigma_\xi|_{\mathcal{R}_F(c_\sigma)}$  is irreducible.

The degree  $[L:F]$  is the number of distinct characters occurring in the representation  $\sigma|_{\mathcal{R}_F^+(c_\sigma)} = \tau|_{\mathcal{R}_T^+(c_\tau)}$ , so  $[L:F] = [LT:T]$  and  $L/F$  is totally wildly ramified. Further,

$$\text{Ind}_{L/F} \sigma_\xi|_{\mathcal{W}_T} = \text{Ind}_{LT/T} (\sigma_\xi|_{\mathcal{W}_{LT}}).$$

This restriction is irreducible, whence  $\text{Ind}_{L/F} \sigma_\xi$  is irreducible and equivalent to  $\sigma$ . Finally, for  $0 \leq x \leq c_\sigma$ ,

$$\Psi_\sigma(x) = \Psi_\tau(ex)/e = p^{-r} \psi_{LT/T}(ex)/e = p^{-r} \psi_{L/F}(x),$$

by 9.2 Complement 1 and 1.1 Lemma.  $\square$

**Complement.** *If  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ , the assertions of 9.2 Complement 1 apply unchanged.*

*Proof.* Immediate.  $\square$

**9.6.** We continue with the notation of 9.5 Corollary.

**Definition.** Let  $\widetilde{L}_{\sigma,\xi}/L_\xi$  to be the centric field of the representation  $\sigma_\xi \in \widehat{\mathcal{W}}_{L_\xi}^{\text{wr}}$ .

The extension  $\widetilde{L}_{\sigma,\xi}/L_\xi$  is Galois and  $\widetilde{L}_{\sigma,\xi}/F$  is uniquely determined by  $\sigma$ , up to isomorphism.

**Proposition.** *Suppose  $\sigma$  is absolutely wildly ramified. The extension  $\widetilde{L}_{\sigma,\xi}/L_\xi$  is totally ramified and elementary abelian of degree  $(\dim \sigma_\xi)^2$ . If  $\widetilde{L}_{\sigma,\xi} \neq L_\xi$ , the extension  $\widetilde{L}_{\sigma,\xi}/L_\xi$  has a unique ramification jump, lying at  $\psi_{L_\xi/F}(c_\alpha)$ . In particular,  $\mathcal{R}_F^+(c_\alpha) \subset \mathcal{W}_{\widetilde{L}_{\sigma,\xi}}$ .*

*Proof.* The representation  $\sigma_\xi$  is absolutely wild. It is irreducible on restriction to  $\mathcal{R}_{L_\xi}(\psi_{L_\xi/F}(c_\alpha)) = \mathcal{R}_F(c_\alpha)$ , while it is a multiple of the character  $\xi$  on  $\mathcal{R}_F^+(c_\alpha)$ . All assertions are now instances of the discussion in section 8, especially 8.3.  $\square$

In the general case  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ , the extension  $\widetilde{L}_{\sigma,\xi}/L_\xi$  is not totally wildly ramified. We recall the standard example.

**Example.** For this example, we adhere to the classical framework of the exposition in section 41 of [8]. Take  $p = 2$ , and let  $\sigma \in \widehat{\mathcal{W}}_F$  be *primitive* of dimension 2. The representation  $\sigma$  is then totally ramified and H-singular. After twisting with a character, if necessary, we may assume that  $\sigma$  is of Carayol type. In terms of the preceding discussion,  $\Psi_\sigma$  has one jump and  $L_\xi = F$ . Using the standard notation for permutation groups,  $\bar{\sigma}(\mathcal{W}_F)$  is either  $A_4$  (if  $F$  has a primitive cube root of unity) or  $S_4$  (otherwise). The tame centric field  $T/F$  is cyclic of degree 3 in the first case and, in the second,  $\text{Gal}(T/F) \cong S_3$ .

**9.7.** As an application of the methods of this section, we return to the question left open in 8.2 Remark. If  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$ , we use the notation  $D(\sigma)$ ,  $D_0(\sigma)$ ,  $T_I(\sigma)$  introduced in section 8. In addition,  $T(\sigma)$  shall be the tame centric field of  $\sigma$ .

**Application.** *There exists a field  $F$ , of residual characteristic 2, and a representation  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  such that  $T_I(\sigma) \neq T(\sigma)$ . One may take  $\sigma$  to be of Carayol type and dimension 4.*

*Proof.* Let  $F$  have residual characteristic 2. Let  $K/F$  be totally ramified of degree 4, such that  $\psi_{K/F}$  has two jumps  $a < b$ , of which  $a$  is an odd integer. Replacing  $F$  by  $E$  and  $K/F$  by  $EK/E$ , where  $E/F$  is finite and tamely ramified, we may assume  $b-a$  to be as large as necessary without affecting the parity of  $a$ .

Let  $m$  be a positive integer and define  $c = c_m$  by the equation  $4c + \psi_{K/F}(c) = m$ .

**Lemma.** *If  $b-a$  is sufficiently large, one may choose  $m$  so that*

- (1)  $a < c_m < b$ ,
- (2)  $m \not\equiv 2a \pmod{3}$ ,
- (3)  $m \equiv a+2 \pmod{4}$

This is clear. Assume it has been done, and note that  $m$  is odd. We get

$$c = c_m = (m-2a)/6.$$

The bi-Herbrand function  $\Psi = {}^2\Psi_{(K/F, m/4)}$  has three jumps, namely  $a$ ,  $c$  and  $z$ , satisfying  $a < c < z$ . By 7.2 Corollary, there exists  $\Theta \in \mathcal{E}^C(F)$  such that  $\Psi(x) = \Psi_\Theta(x)$ ,  $0 \leq x \leq m/4$ . Choose  $\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}}$  such that  $[\sigma]_0^+ = {}^L\Theta$ . We show that  $\sigma$  has the desired properties.

Let  $\phi \in D_0(\sigma)$ . The  $F$ -slope  $\text{sl}_F(\phi)$  of  $\phi$ , as in (8.3.1), can only take a value  $0, a, c, z$  (cf. 8.1 Proposition of [13]). If  $\text{sl}_F(\phi) = 0$ , then  $\phi$  corresponds (via 8.2 Lemma) to the trivial element of  $D(\sigma)$ . Suppose  $\text{sl}_F(\phi) = a$ . Since  $a$  is an integer, the  $\mathcal{W}_F$ -stabilizer of  $\phi|_{\mathcal{R}_F(a)}$  is of the form  $\mathcal{W}_E$ , where  $E/F$  is unramified. The character  $\phi \in D_0(\sigma)$  is completely determined by its restriction to  $\mathcal{R}_F(a)$ . Consequently,  $\mathcal{W}_E$  is the  $\mathcal{W}_F$ -stabilizer of  $\phi$ . So, writing  $\sigma_E = \sigma|_{\mathcal{W}_E}$ , there exists a unique character  $\tilde{\phi} \in D(\sigma_E)$  such that  $\tilde{\phi}|_{\mathcal{P}_F} = \phi$  (8.2 Lemma).

Suppose next that  $\text{sl}_F(\phi) = c = (m-2a)/6$ . The conditions imposed on  $m$  imply  $3c \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ . We conclude that there is no finite tame extension  $E/F$  for which  $\phi$  extends to a character of  $\mathcal{W}_E$ . Finally, consider the case where  $\text{sl}_F(\phi) = z$ . By 3.5 Theorem,  $z = (m-a)/4 \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$  and the same conclusion holds. We have shown:

**Proposition.** *The group  $D_0(\sigma)$  has order 2 and  $T_I(\sigma)/F$  is unramified.*

We now follow the procedure of 9.5 to choose a character  $\xi$  of  $\mathcal{R}_F^+(c)$  occurring in  $\sigma|_{\mathcal{R}_F^+(c)}$ . We set  $L = L_\xi$  and  $\tau = \sigma_\xi$ . We have  $\sigma = \text{Ind}_{L/F} \tau$ . Since

$\text{sw}(\sigma) = m$  and  $w_{L/F} = a$ , we get  $\text{sw}(\tau) = m - 2a \not\equiv 0 \pmod{3}$ . The Herbrand function  $\Psi_\tau$  has a unique jump, which lies at  $(m - 2a)/3$  (8.3 Proposition 2). It follows that  $e(T_I(\tau)|L)$  is divisible by 3. This implies that  $e(T_\sigma|F)$  is divisible by 3, whence  $T(\sigma) \neq T_I(\sigma)$ .  $\square$

*Remarks.*

- (1) We could have treated the classical example of 9.6 within the framework of the preceding paragraph.
- (2) While we have phrased the argument as an existence statement, it may be refined to produce a specific example.
- (3) We imposed the restriction  $p = 2$  to keep the argument simple: it works equally well for any prime  $p$  starting from an extension  $K/F$  of degree  $p^2$  with two ramification jumps.

## 10. Parameter fields

Let  $[\mathfrak{a}, m, 0, \alpha]$  be a simple stratum in  $M_{p^r}(F)$ ,  $r \geq 1$ , with the usual properties:  $F[\alpha]/F$  is totally ramified of degree  $p^r$  and  $m = -v_{F[\alpha]}(\alpha)$  is not divisible by  $p$ . Let

$$\mathcal{G}^*(\alpha) = \{\sigma \in \widehat{\mathcal{W}}_F^{\text{wr}} : [\sigma]_0^+ \in {}^L\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|\}.$$

Take  $\sigma \in \mathcal{G}^*(\alpha)$ , with  $[\sigma]_0^+ = {}^L\Theta$ . We have two determinations of  $\Psi_\Theta$ , from 7.2 Theorem 2 and 9.5 Corollary. As in 9.5, 9.6, we attach to  $\sigma$  the tower of fields  $F \subset L_\xi \subset \widetilde{L}_{\sigma, \xi}$ , given by a character  $\xi$  of  $\mathcal{R}_F^+(c_\alpha)$  occurring in  $\sigma$ . In particular, this tower of fields is determined by  $\sigma$  up to  $\mathcal{W}_F$ -conjugation. We examine how it varies as  $\Theta$  ranges over  $\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ .

The relation between the pair of fields  $L_\xi \subset \widetilde{L}_{\sigma, \xi}$  and the automorphic parameter field  $F[\alpha]$  is a more complex matter, beyond the scope of this paper. We comment briefly on its history at the end of the section.

**10.1.** We fix notation for the rest of the section. With  $[\mathfrak{a}, m, 0, \alpha]$  as above, we abbreviate

$$(10.1.1) \quad \begin{aligned} \varsigma_\alpha &= m/p^r, & w_\alpha &= w_{F[\alpha]/F}, \\ l_\alpha &= \max(0, m - w_\alpha), & \lambda_\alpha &= [l_\alpha/2]. \end{aligned}$$

By 7.2 Theorem 2,  $\Psi_\Theta(x) = {}^2\Psi_{(F[\alpha]/F, \varsigma_\alpha)}(x)$ ,  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ ,  $0 \leq x \leq \varsigma_\alpha$ . We use the notation

$$(10.1.2) \quad \begin{aligned} {}^2\Psi_{(F[\alpha]/F, \varsigma_\alpha)} &= {}^2\Psi_\alpha, \\ c_\alpha + {}^2\Psi_\alpha(c_\alpha) &= \varsigma_\alpha, \\ {}^2\Psi_\alpha(\epsilon_\alpha) &= \lambda_\alpha/p^r. \end{aligned}$$

Let  $\mathcal{G}_0^*(\alpha)$  be the subset  ${}^L\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$  of  $\mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$ . Every element of  $\mathcal{G}_0^*(\alpha)$  is a singleton orbit, so we may treat such orbits as representations of  $\mathcal{P}_F$ . Restriction to  $\mathcal{P}_F$  gives a surjective map  $\mathcal{G}^*(\alpha) \rightarrow \mathcal{G}_0^*(\alpha)$ . Each fibre of this map is a principal homogeneous space over the group of tamely ramified characters of  $\mathcal{W}_F$ , as in [12] 1.3 Proposition.

**10.2.** We give a relative characterization of the elements of  $\mathcal{G}^*(\alpha)$  in terms of the ultrametric pairing  $\Delta$  on  $\widehat{\mathcal{W}}_F$ .

**Proposition.** *Let  $\sigma \in \mathcal{G}^*(\alpha)$ ,  $\tau \in \widehat{\mathcal{W}}_F^{\text{wr}}$ . The following conditions are equivalent:*

- (1)  $\tau \in \mathcal{G}^*(\alpha)$ ;
- (2)  $\dim \tau \leq p^r$  and  $\Delta(\sigma, \tau) \leq \epsilon_\alpha$ ;
- (3)  $\dim \tau \leq p^r$  and  $\text{Hom}_{\mathcal{R}_F^+(\epsilon_\alpha)}(\sigma, \tau) \neq 0$ .

*Proof.* We work first on the GL-side.

**Lemma.** *Let  $\Theta \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$  and  $\Xi \in \mathcal{E}(F)$ . The following are equivalent:*

- (1)  $\Xi \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ ;
- (2)  $\deg \Xi \leq p^r$  and  $\mathbb{A}(\Xi, \Theta) \leq \lambda_\alpha/p^r$ .

*Proof.* Let  $\theta \in \mathcal{C}^*(\mathfrak{a}, \alpha)$  have endo-class  $\Theta$ . If  $\Xi \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ , then  $\deg \Xi = p^r$  and  $\Xi$  is the endo-class of some  $\phi \in \mathcal{C}^*(\mathfrak{a}, \alpha)$ . By definition,  $\phi$  agrees with  $\theta$  on  $H^{1+\lambda_\alpha}(\alpha, \mathfrak{a})$ , whence  $\mathbb{A}(\Xi, \Theta) \leq \lambda_\alpha/p^r$ . Thus (1) implies (2).

Assume (2) holds. Since  $\mathbb{A}(\Xi, \Theta) \leq l_\alpha/2p^r < m/p^r$ , we conclude that  $\varsigma_\Xi = m/p^r$ : this follows from the definition of  $\mathbb{A}$ . As  $\deg \Xi \leq p^r$  and  $p$  does not divide  $m$ , so  $\deg \Xi = p^r$  and  $\Xi$  has a realization  $\phi \in \mathcal{C}(\mathfrak{a}, \beta)$ , for a simple stratum  $[\mathfrak{a}, m, 0, \beta]$  in which  $F[\beta]/F$  is totally ramified of degree  $p^r$ . The characters  $\phi|_{H^{1+\lambda_\alpha}(\beta, \mathfrak{a})}$ ,  $\theta|_{H^{1+\lambda_\alpha}(\alpha, \mathfrak{a})}$  intertwine in  $\text{GL}_{p^r}(F)$  by hypothesis. Since  $\lambda_\alpha < m/2$ , (3.5.11) Theorem of [15] allows us to replace  $\phi$  by a conjugate to achieve  $H^1(\beta, \mathfrak{a}) = H^1(\alpha, \mathfrak{a})$  and  $\phi \in \mathcal{C}(\mathfrak{a}, \alpha)$ . The characters  $\phi|_{H^{1+\lambda_\alpha}(\alpha, \mathfrak{a})}$ ,  $\theta|_{H^{1+\lambda_\alpha}(\alpha, \mathfrak{a})}$  intertwine and so are equal [15] (3.3.2). That is,  $\phi \in \mathcal{C}^*(\mathfrak{a}, \alpha)$ , as required.  $\square$

In the proposition, write  $[\sigma]_0^+ = {}^L\Theta$ ,  $[\tau]_0^+ = {}^L\Xi$ . In particular,  $\Theta \in \mathcal{E}^C(F)$  while  $\Xi \in \mathcal{E}(F)$  is totally wild of degree at most  $p^r$ . We have  $\Psi_\Theta(\Delta(\sigma, \tau)) = \mathbb{A}(\Theta, \Xi)$ . The definition (10.1.2) shows that  $\mathbb{A}(\Theta, \Xi) \leq \lambda_\alpha/p^r$  if and only if  $\Delta(\sigma, \tau) \leq \epsilon_\alpha$ . The proposition thus follows from the lemma.  $\square$

*Remark.* In the lemma, the hypothesis  $\deg \Xi \leq p^r$  is essential. For, the Density Lemma of [13] 5.3 shows that the set of values  $\mathbb{A}(\Theta, \Xi)$ ,  $\Xi \in \mathfrak{E}(F)$ , is dense on the positive real axis. Indeed, the proof there shows that the result remains valid, for this  $\Theta$ , when  $\Xi$  is confined to the set of totally wild endo-classes. In the proposition, the hypothesis  $\dim \tau \leq p^r$  is likewise essential. Interpretation of the general case, with  $\dim \tau$  unbounded, is the subject of [13] 6.5 Corollary.

**10.3.** We continue with the notation of (10.1.1) and (10.1.2). In addition,  $j_\infty(\alpha)$  will mean  $j_\infty(F[\alpha]|F)$ , the greatest jump of the function  $\psi_{F[\alpha]/F}$ . We prove:

**Theorem.**

- (1) *There is a character  $\xi$  of  $\mathcal{R}_F^\perp(c_\alpha)$  occurring in every representation  $\sigma \in \mathcal{G}^*(\alpha)$ . This condition determines  $\xi$  uniquely, up to  $\mathcal{W}_F$ -conjugation. In particular, each  $\sigma \in \mathcal{G}^*(\alpha)$  determines the same conjugacy class of field extensions  $L_\xi/F$ .*
- (2) *Suppose that either  $j_\infty(\alpha) > c_\alpha$  or that  $l_\alpha$  is odd. There is then an irreducible representation  $\rho_\xi$  of  $\mathcal{R}_F(c_\alpha)$  occurring in every  $\sigma \in \mathcal{G}^*(\alpha)$ . This condition determines  $\rho_\xi$  uniquely, up to  $\mathcal{W}_F$ -conjugation.*

*Proof.* We estimate the number  $c_\alpha$  to get a more precise bound for the distance  $\Delta(\sigma_1, \sigma_2)$ ,  $\sigma_i \in \mathcal{G}^*(\alpha)$ .

**Lemma.** *Write  $j_\infty(\alpha) = j_\infty(F[\alpha]|F)$ .*

- (1) *If  $j_\infty(\alpha) \leq c_\alpha$ , then  $c_\alpha = (m+w_\alpha)/2p^r$ .*
- (2) *If  $j_\infty(\alpha) > c_\alpha$ , then  $c_\alpha < (m+w_\alpha)/2p^r$ .*

*Proof.* Suppose  $j_\infty(\alpha) < c_\alpha$ . The function  ${}^2\Psi_\alpha$  then has an even number of jumps, its graph contains a non-empty open segment of the line  $y = x - p^{-r}w_\alpha$ , and  $x = c_\alpha$  is the intersection of this line segment with  $x+y = \varsigma_\alpha$  (4.2 Proposition). That is,  $c_\alpha = (m+w_\alpha)/2p^r$ .

Suppose next that  $j_\infty(\alpha) = c_\alpha$ . Therefore  ${}^2\Psi_\alpha(c_\alpha) = p^{-r}\psi_{F[\alpha]/F}(c_\alpha) = c_\alpha - p^{-r}w_\alpha$ . Thus  $2c_\alpha - p^{-r}w_\alpha = \varsigma_\alpha$  and  $c_\alpha = (m+w_\alpha)/2p^r$ , as desired.

In (2), the line  $y = x - p^{-r}w_\alpha$  lies strictly below the graph  $y = {}^2\Psi_\alpha(x)$ , (cf. 1.6 Proposition, 4.2 Proposition), giving

$$\varsigma_\alpha - c_\alpha = {}^2\Psi_\alpha(c_\alpha) > c_\alpha - p^{-r}w_\alpha,$$

and hence the result.  $\square$

**Proposition.** *If  $\sigma_1, \sigma_2 \in \mathcal{G}^*(\alpha)$  and  $[\sigma_i]_0^+ = {}^L\Theta_i$ , then*

- (1)  $\Delta(\sigma_1, \sigma_2) \leq c_\alpha$ ;
- (2)  $\Delta(\sigma_1, \sigma_2) = c_\alpha$  if and only if
  - (a)  $\mathbb{A}(\Theta_1, \Theta_2) = l_\alpha/2p^r$  and
  - (b)  $j_\infty(\alpha) \leq c_\alpha$ .

*Proof.* We have  $\mathbb{A}(\Theta_1, \Theta_2) = {}^2\Psi_\alpha(\Delta(\sigma_1, \sigma_2))$ . Defining  $\epsilon_\alpha$  by (10.1.1), 10.2 Proposition gives  $\Delta(\sigma_1, \sigma_2) \leq \epsilon_\alpha$ , with equality if and only if  $\mathbb{A}(\Theta_1, \Theta_2) = l_\alpha/p^r$ .

Following the lemma, we distinguish two cases. Suppose first that  $j_\infty(\alpha) \leq c_\alpha$ , whence  $c_\alpha = (m+w_\alpha)/2p^r$ . Here,

$${}^2\Psi_\alpha(c_\alpha) = \varsigma_\alpha - c_\alpha = m/p^r - (m+w_\alpha)/2p^r = (m-w_\alpha)/2p^r = l_\alpha/2p^r.$$

That is,  $\epsilon_\alpha \leq c_\alpha$ , with equality if and only if  $l_\alpha$  is even. In this case, therefore,  $\Delta(\sigma_1, \sigma_2) \leq c_\alpha$ , with equality if and only if  $\mathbb{A}(\Theta_1, \Theta_2) = l_\alpha/2p^r$ .

Take now the case  $j_\infty(\alpha) > c_\alpha$ . As in the lemma,  $c_\alpha < (m+w_\alpha)/2p^r$  and the line  $y = x - p^{-r}w_\alpha$  lies strictly below the graph  $y = {}^2\Psi_\alpha(x)$ . The three lines  $y = l_\alpha/2p^r$ ,  $y = x - p^{-r}w_\alpha$  and  $x+y = \varsigma_\alpha$  all meet at  $x = (m+w_\alpha)/2p^r$ . By the lemma,  $(m+w_\alpha)/2p^r > c_\alpha$ , so  ${}^2\Psi_\alpha(c_\alpha) > l_\alpha/2p^r$ . It follows that  $\Delta(\sigma_1, \sigma_2) \leq \epsilon_\alpha < c_\alpha$ , as required.  $\square$

*Remark.* If  $\Theta_1, \Theta_2 \in \|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ , then  $p^r \mathbb{A}(\Theta_1, \Theta_2)$  is an integer: this follows readily from the definition of  $\mathbb{A}$ . So, condition (a) in part (2) of the proposition can only hold when  $l_\alpha$  is even.

We now prove the theorem. Choose  $\sigma \in \mathcal{G}^*(\alpha)$  and apply 9.5 Corollary. In the notation of that result,  $\sigma|_{\mathcal{R}_F^+(c_\alpha)}$  is a direct sum of  $\mathcal{W}_F$ -conjugate characters  $\xi$ . If  $\tau \in \mathcal{G}^*(\alpha)$ , the proposition gives  $\Delta(\sigma, \tau) \leq c_\alpha$  whence any such  $\xi$  occurs in  $\tau$ . The uniqueness property follows by symmetry.

With the same  $\sigma$  and  $\xi$ , define  $L = L_\xi$  so that  $\mathcal{W}_L$  is the  $\mathcal{W}_F$ -stabilizer of  $\xi$ . In particular,  $\mathcal{R}_F(c_\alpha) \subset \mathcal{W}_L$ . Let  $\sigma_\xi$  be the natural representation of  $\sigma$  on the  $\xi$ -isotypic subspace of  $\sigma$ . Thus  $\sigma_\xi$  is either one-dimensional or H-singular (9.5 Corollary). In either case, the representation  $\rho_\xi = \sigma_\xi|_{\mathcal{R}_F(c_\alpha)}$  is irreducible. By hypothesis,  $\Delta(\sigma, \tau) < c_\alpha$ ,  $\tau \in \mathcal{G}^*(\alpha)$ . The representation  $\rho_\xi$  thus has the required properties.  $\square$

We underline some useful facts that emerged during the preceding arguments.



**Corollary.**

- (1) If  $j_\infty(\alpha) \leq c_\alpha$  then  ${}^2\Psi_\alpha(c_\alpha) = l_\alpha/2p^r$ .
- (2) If  $j_\infty(\alpha) > c_\alpha$ , then  ${}^2\Psi_\alpha(c_\alpha) > l_\alpha/2p^r \geq \lambda_\alpha/p^r$ .

**10.4.** We fix a character  $\xi$  of  $\mathcal{R}_F^+(c_\alpha)$ , as in 10.3 Theorem, and let  $\mathcal{W}_L$  be the  $\mathcal{W}_F$ -stabilizer of  $\xi$ . If  $\sigma \in \mathcal{G}^*(\alpha)$ , then  $\sigma_\xi$  denotes the natural representation of  $\mathcal{W}_L$  on the  $\xi$ -isotypic subspace of  $\sigma$ . We write  $\tilde{L}_\sigma = \tilde{L}_{\sigma, \xi}/L$  for the centric field of the H-singular representation  $\sigma_\xi$ ,  $\sigma \in \mathcal{G}^*(\alpha)$ , as in 9.6.

**Theorem.** *If either  $l_\alpha$  is odd or  $j_\infty(\alpha) \neq c_\alpha$ , then  $\tilde{L}_\sigma = \tilde{L}_\tau$ , for all  $\sigma, \tau \in \mathcal{G}^*(\alpha)$ .*

*Proof.* We start with a general result, requiring no restriction on  $j_\infty(\alpha)$  or  $l_\alpha$ . Let  $\Delta_L$  be the canonical ultrametric pairing on  $\hat{\mathcal{W}}_L$ .

**Lemma.** *If  $\sigma, \sigma' \in \mathcal{G}^*(\alpha)$ , then*

$$(10.4.1) \quad \max \{ \Delta_L(\sigma_\xi, \sigma'_\xi) : \sigma, \sigma' \in \mathcal{G}^*(\alpha) \} = \lambda_\alpha.$$

*Proof.* By 10.2 Lemma,  $\max \{ \mathbb{A}(\Theta, \Theta') : \Theta, \Theta' \in \|\mathcal{C}^*(\mathbf{a}, \alpha)\| \} = \lambda_\alpha/p^r$ . So,

$$\max \{ \Delta(\sigma, \sigma') : \sigma, \sigma' \in \mathcal{G}^*(\alpha) \} = {}^2\Psi_\alpha^{-1}(\lambda_\alpha/p^r) = \varphi_{L/F}(\lambda_\alpha),$$

by (9.5.1). The relation (10.4.1) now follows from 1.4 Proposition.  $\square$

Let  $k \geq 0$  be an integer and  $K/F$  a finite field extension. Let  $\Gamma_k(K)$  be the group of characters of  $K^\times/U_K^{1+k}$ , and  $\Gamma_k^0(K)$  the group of characters of  $U_K^1/U_K^{1+k}$ .

Let  $\mathcal{H}(L, \xi)$  be the set of representations  $\sigma_\xi$ ,  $\sigma \in \mathcal{G}^*(\alpha)$ . The induction functor  $\text{Ind}_{L/F}$  then gives a bijection  $\mathcal{H}(L, \xi) \rightarrow \mathcal{G}^*(\alpha)$ . Similarly, let  $\mathcal{H}_0(L, \xi)$  be the set of representations  $\sigma_\xi|_{\mathcal{P}_L}$ ,  $\sigma \in \mathcal{G}^*(\alpha)$ . Induction, from  $\mathcal{P}_L$  to  $\mathcal{P}_F$ , gives a bijection  $\mathcal{H}_0(L, \xi) \rightarrow \mathcal{G}_0^*(\alpha)$ .

**Proposition.**

- (1) If  $\chi \in \Gamma_{\lambda_\alpha}(L)$  and  $\kappa \in \mathcal{H}(L, \xi)$ , then  $\chi \otimes \kappa \in \mathcal{H}(L, \xi)$ .
- (2) Suppose that either  $j_\infty(\alpha) \neq c_\alpha$  or that  $l_\alpha$  is odd. The set  $\mathcal{H}(L, \xi)$  is then a principal homogeneous space over  $\Gamma_{\lambda_\alpha}(L)$ .

*Proof.* In (1), the representations  $\kappa$ ,  $\chi \otimes \kappa$  agree on the group  $\mathcal{R}_L^+(\lambda_\alpha) = \mathcal{W}_L \cap \mathcal{R}_F^+(\varphi_{L/F}(\lambda_\alpha)) = \mathcal{W}_L \cap \mathcal{R}_F^+(\epsilon_\alpha)$ . Each induces irreducibly to  $\mathcal{W}_F$ , so they agree on  $\mathcal{R}_F(\epsilon_\alpha)$  whence  $\chi \otimes \kappa \in \mathcal{H}(L, \xi)$  by 10.2 Proposition.

In the second part, it is enough to prove that  $\mathcal{H}_0(L, \xi)$  is a principal homogeneous space over  $\Gamma_{\lambda_\alpha}^0(L, \xi)$ . The sets  $\mathcal{H}_0(L, \xi)$ ,  $\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ ,  $\mathcal{C}^*(\mathfrak{a}, \alpha)$  are in canonical bijection, and  $\mathcal{C}^*(\mathfrak{a}, \alpha)$  has exactly  $q^{\lambda_\alpha}$  elements, where  $q$  is the cardinality of the residue field of  $F$ . This reduces us to showing that, if  $\kappa \in \mathcal{H}_0(L, \xi)$  and  $\chi \in \Gamma_{\lambda_\alpha}^0(L)$ ,  $\chi \neq 1$ , then  $\chi \otimes \kappa \not\cong \kappa$ .

If  $j_\infty(\alpha) < c_\alpha$ , the representation  $\kappa$  is a character, and the result is obvious. Suppose that  $j_\infty(\alpha) = c_\alpha$  and  $l_\alpha$  is odd, or that  $j_\infty(\alpha) > c_\alpha$ . In either case, the function  ${}^2\Psi_\alpha$  has an odd number of jumps. It follows from 9.2 Complement 2 that  $j_\infty(L|F) < j_\infty(\alpha)$  and so  $\mathcal{R}_F(c_\alpha) \subset \mathcal{W}_L$ . The restriction  $\kappa|_{\mathcal{R}_F(c_\alpha)}$  is irreducible. If  $\rho$  is a representation of  $\mathcal{P}_L$  such that  $\rho|_{\mathcal{R}_F(c_\alpha)} = \kappa|_{\mathcal{R}_F(c_\alpha)}$ , there is a *unique* character  $\phi$  of  $\mathcal{P}_L$ , trivial on  $\mathcal{R}_F(c_\alpha)$ , such that  $\rho = \phi \otimes \kappa$ . By 10.2 Corollary, any  $\chi \in \Gamma_{\lambda_\alpha}^0(L)$  is trivial on  $\mathcal{R}_F(c_\alpha)$ . The representations  $\chi \otimes \kappa$  are therefore distinct, as  $\chi$  ranges over  $(U_L^1/U_L^{1+\lambda-\alpha})^\wedge$ , and the proposition follows.  $\square$

We prove the theorem. If  $j_\infty < c_\alpha$ , then  $\tilde{L}_\sigma = \tilde{L}_\tau = L$ . In all other cases, we have  $\tau_\xi = \chi \otimes \sigma_\xi$ , for some  $\chi \in \Gamma_{\lambda_\alpha}(L)$ . The representations  $\tau_\xi, \sigma_\xi$  therefore define the same projective representation of  $\mathcal{W}_L$  whence follows the theorem.  $\square$

*Remarks.*

- (1) In the exceptional case of  $j_\infty(\alpha) = c_\alpha$  and  $l_\alpha \equiv 0 \pmod{2}$ , there may exist non-trivial characters  $\chi \in \Gamma_{\lambda_\alpha}(L)$  such that  $\chi \otimes \sigma_\xi \cong \sigma_\xi$ . This is incompatible with a principal homogeneous space structure.
- (2) This case occurs quite commonly, the first being where  $p^r = 2$ ,  $w_\alpha$  is odd (cf. 1.8), and  $m = 3w_\alpha$ .
- (3) Consider the case  $l_\alpha = 0$ . Thus  $\lambda_\alpha = 0$  and the sets  $\|\mathcal{C}^*(\mathfrak{a}, \alpha)\|$ ,  $\mathcal{G}_0^*(\alpha)$  each have only one element. The conclusions of the theorem and proposition then hold trivially. In this case, we have  $w_\alpha \geq m$  and  $j_\infty(\alpha) \geq w_\alpha/(p^r - 1) > c_\alpha$ .

**10.5.** It remains to treat the case where  $j_\infty(\alpha) = c_\alpha$  and  $l_\alpha \equiv 0 \pmod{2}$ . In light of 10.4 Remark (3), we may assume  $l_\alpha > 0$ . Let  $T_\sigma/L$  be the maximal tame sub-extension of  $\tilde{L}_\sigma/L$ , and define the character group  $D(\sigma_\xi)$  as in 8.2.

**Theorem.** *Suppose that  $j_\infty(\alpha) = c_\alpha$  and  $l_\alpha \equiv 0 \pmod{2}$ ,  $l_\alpha > 0$ .*

- (1) *If  $\sigma, \tau \in \mathcal{G}^*(\alpha)$ , then  $T_\sigma = T_\tau$ .*
- (2) *The integer  $d = |D(\sigma_\xi)|$  is independent of the choice of  $\sigma \in \mathcal{G}^*(\alpha)$ . It satisfies  $d^{\frac{1}{2}} \leq \dim \sigma_\xi = p^r/[L:F]$ .*

- (3) *There are, at most,  $d$  distinct Galois extensions of the form  $\tilde{L}_\sigma/L$ , as  $\sigma$  ranges over  $\mathcal{G}^*(\alpha)$ . If  $p$  does not divide  $[T_\sigma:L]$ , there are exactly  $d$  such extensions.*

*Proof.* We gather some identities. First,  ${}^2\Psi_\alpha(x) = p^{-r}\psi_{L/F}(x)$ ,  $0 \leq x \leq c_\alpha$ , by 9.5 Corollary. Since  $j_\infty(\alpha) = c_\alpha$ , 10.2 Corollary gives  ${}^2\Psi_\alpha(c_\alpha) = l_\alpha/2p^r$ . Consequently

$$(10.5.1) \quad \psi_{L/F}(c_\alpha) = l_\alpha/2.$$

In this situation, (9.5.1) implies  $j_\infty(\alpha) > j_\infty(L|F)$ , so

$$(10.5.2) \quad \mathcal{R}_F(c_\alpha) = \mathcal{R}_L(\psi_{L/F}(c_\alpha)) = \mathcal{R}_L(l_\alpha/2).$$

Write  $e_\sigma = e(T_\sigma|L)$ , so that  $\mathcal{R}_L(l_\alpha/2) = \mathcal{R}_{T_\sigma}(e_\sigma l_\alpha/2)$ . The point  $e_\sigma l_\alpha/2$  is the unique jump of  $\tilde{L}_\sigma/T_\sigma$ , so

$$(10.5.3) \quad \mathcal{R}_F^+(c_\alpha) = \mathcal{R}_L^+(\psi_{L/F}(c_\alpha)) = \mathcal{R}_{\tilde{L}_\sigma}^+(e_\sigma l_\alpha/2),$$

and

$$(10.5.4) \quad \mathcal{W}_{T_\sigma} = \mathcal{W}_{\tilde{L}_\sigma} \mathcal{R}_L(l_\alpha/2).$$

**Lemma 1.** *If  $\sigma, \tau \in \mathcal{G}^*(\alpha)$ , then  $T_\tau = T_\sigma$ .*

*Proof.* The group  $\mathcal{W}_{T_\sigma}$  is the  $\mathcal{W}_L$ -centralizer of  $\sigma_\xi(\mathcal{R}_L(\psi_{L/F}(c_\alpha)))$  modulo its centre (8.2 Proposition). This centre, we assert, is independent of  $\sigma$ . The pairing  $(x, y) \mapsto \xi([x, y])$  defines an alternating form on the  $\mathbb{F}_p$ -vector space  $\mathcal{R}_L(l_\alpha/2)/\mathcal{R}_L^+(l_\alpha/2)$ . Let  $R$  be the inverse image, in  $\mathcal{R}_L(l_\alpha/2)$ , of the radical of this pairing. Since  $\mathcal{W}_L$  fixes  $\xi$ , it normalizes  $R$ . The image  $\sigma_\xi(R)$  is the centre of  $\sigma_\xi(\mathcal{R}_L(l_\alpha/2))$ . Thus  $\mathcal{W}_{T_\sigma}$  is the  $\mathcal{W}_L$ -centralizer of the finite group  $\mathcal{R}_L(l_\alpha/2)/R$  and so is independent of  $\sigma$ .  $\square$

Lemma 1 is part (1) of the theorem. The integer  $\dim \sigma_\xi = p^r/[L:F]$  is certainly independent of  $\sigma \in \mathcal{G}^*(\alpha)$ . By 8.2 Lemma (1), the order of the group  $D(\sigma_\xi)$  is the number of fixed points for the natural action of  $\mathcal{W}_{T_\sigma}$  on  $\mathcal{R}_L(l_\alpha/2)/R$ , in the notation of the proof of Lemma 1. It is therefore independent of  $\sigma$  and we have proved part (2) of the theorem.

In light of part (1), we abbreviate  $T = T_\sigma$  and  $e = e(T|L)$ .

**Lemma 2.** *Suppose that  $T = L$ . For  $\tau \in \mathcal{G}^*(\alpha)$ , the following are equivalent:*

- (1)  $\tilde{L}_\tau = \tilde{L}_\sigma$ ;
- (2) *there is a character  $\chi$  of  $\mathcal{W}_L$ , trivial on  $\mathcal{R}_L^+(l_\alpha/2)$ , such that  $\tau_\xi \cong \chi \otimes \sigma_\xi$ .*

*Proof.* Surely (2) implies (1), so suppose that (1) holds. The restrictions  $\sigma'_\xi = \sigma_\xi|_{\mathcal{R}_L(l_\alpha/2)}$ ,  $\tau'_\xi = \tau_\xi|_{\mathcal{R}_L(l_\alpha/2)}$  are irreducible, and each is a multiple of  $\xi$  on  $\mathcal{R}_L^+(l_\alpha/2)$ . On the group  $R$  (as in the proof of Lemma 1), each is a multiple of a character of  $R$  extending  $\xi$ . Consequently, there is a character  $\phi_R$  of  $R$ , trivial on  $\mathcal{R}_L^+(l_\alpha/2)$ , such that  $\tau_\xi|_R = \phi_R \otimes \sigma_\xi|_R$ . The character  $\phi_R$  extends to a character  $\phi$  of  $\mathcal{R}_L(l_\alpha/2)$ . For any such  $\phi$ , we have  $\tau'_\xi = \phi \otimes \sigma'_\xi$ . The projective representations  $\bar{\sigma}_\xi, \bar{\tau}_\xi$  defined by  $\sigma_\xi, \tau_\xi$  are therefore identical on  $\mathcal{R}_L(l_\alpha/2)$ . Each of these projective representations has  $\mathcal{W}_{\tilde{L}_\sigma} = \mathcal{W}_{\tilde{L}_\tau}$  in its kernel, so  $\bar{\sigma}_\xi, \bar{\tau}_\xi$  are the same on the group  $\mathcal{W}_L = \mathcal{W}_T = \mathcal{W}_{\tilde{L}_\sigma} \mathcal{R}_L(l_\alpha/2)$ . That is,  $\sigma_\xi, \tau_\xi$  are liftings to  $\mathcal{W}_L$  of the same projective representation  $\bar{\sigma}_\xi$ . It follows that  $\tau_\xi \cong \chi \otimes \sigma_\xi$ , for some character  $\chi$  of  $\mathcal{W}_L$  trivial on  $\mathcal{R}_L^+(l_\alpha/2)$ .  $\square$

In the case  $T = L$ , we have  $D(\sigma_\xi) \subset \Gamma_{l_\alpha/2}(L)$ , so Lemma 2 implies that the number of distinct fields  $\tilde{L}_\sigma/L$ ,  $\sigma \in \mathcal{G}^*(\alpha)$ , is

$$|\Gamma_{l_\alpha/2}(L) \backslash \mathcal{G}^*(\alpha)| = |\Gamma_{l_\alpha/2}^0(L) \backslash \mathcal{G}_0^*(\alpha)| = d,$$

as required for part (3) of the theorem.

Return to the general case and write  $e = e(T|L)$ . For  $\sigma \in \mathcal{G}^*(\alpha)$ , write  $\sigma_\xi^T = \sigma_\xi|_{\mathcal{W}_T}$ . Thus  $\sigma_\xi^T$  has centric field  $\tilde{L}_\sigma/T$ . For  $\sigma, \tau \in \mathcal{G}^*(\alpha)$ , Lemma 2 shows that  $\tilde{L}_\sigma = \tilde{L}_\tau$  if and only if there exists  $\chi \in \Gamma_{el_\alpha/2}(T)$  such that  $\tau_\xi^T = \chi \otimes \sigma_\xi^T$ . This condition will certainly hold if  $\tau_\xi = \phi \otimes \sigma_\xi$  for some  $\phi \in \Gamma_{l_\alpha/2}(L)$ . We have proved the first assertion of part (3) of the theorem.

In general, the relation  $\tau_\xi^T = \chi \otimes \sigma_\xi^T$  implies  $\chi/\chi^\gamma \in D(\sigma_\xi^T)$ , for all  $\gamma \in \text{Gal}(T/L)$ . That is,  $\chi$  defines a  $\text{Gal}(T/L)$ -fixed point in  $\Gamma_{el_\alpha/2}(T)/D(\sigma_\xi^T)$ . If  $p$  does not divide  $[T:L]$ , this is equivalent to  $\chi \in \Gamma_{l_\alpha/2}(L)D(\sigma_\xi^T)$ , whence the final assertion follows.  $\square$

*Remark.* There are indeed cases of  $p$  dividing  $[T_\sigma:L]$  in the context of the theorem: we have already seen this in the examples of 9.6, 9.7.

**10.6.** Explicit results concerning the local Langlands correspondence fall into three areas. For essentially tame representations (which have trivial Herbrand functions), complete results are given in [6, 7, 9]. An explicit method for reducing to the totally wild case is worked out in [12]. For totally wildly ramified

representations, results are confined to a small number of old, but very distinguished, papers.

Leaving aside the peripheral situation of [11], all serious work concerns dimension  $p$  in the context of proving the existence of the Langlands correspondence. Kutzko's treatment of the case  $p = 2$  in [31, 32], as given in [8], remains of particular interest. The case  $p = 3$  is in [21], and the extension to  $p \geq 5$  is carried out by Mœglin in [35], building on Kutzko-Moy [33] and Kutzko [30] along with Carayol [17]. Mœglin's paper also relies on a number of working hypotheses since verified, notably

- (1) characterization of the Langlands correspondence via local constants of pairs (see [23]);
- (2) compatibility of Kazhdan's lift [29] and the Kutzko-Moy tame lift [33] with Arthur-Clozel base change [1] (see [25], [4] respectively).

All of those papers assume  $F$  to be of characteristic zero. That restriction is removed in [26, 27].

A concern of the earlier papers was the relation between the fields we have called  $F[\alpha]$ ,  $L$  and  $\tilde{L}_\sigma$  in this section. The case  $p = 2$  is particularly difficult, while it is much more straightforward when  $p \geq 5$ . However, the extreme case of [11], for arbitrary dimension  $p^r$ , shows that the field  $F[\alpha]$  may be so ill-defined as to render the question meaningless without some qualification. On the other hand, suppose  $p \geq 5$  and that  $\sigma = \text{Ind}_{L/F} \chi$  is totally wild of Carayol type, of dimension  $p$ , and that  $L/F$  is *cyclic*. In that case, one may take (in our notation)  $F[\alpha] = L$  [35].

It is quite striking that many phenomena we have encountered are already visible in dimension 2. This applies particularly to the overlapping division into cases of 5.3 Theorem. Part (1) of 10.5 Theorem is a key step in that case: Kutzko shows  $T_\sigma$  to be the splitting field of the polynomial  $X^3 - \text{tr}(\alpha)X^2 + \det(\alpha)$  (45.2 Theorem of [8]). This is generalized to dimension  $p$  in [35] V.4 Proposition. A similar “universal polynomial” appears in [11] 5.1 Theorem for epipelagic representations in arbitrary dimension.

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